DIFFUSION LIMITS OF THE RANDOM WALK METROPOLIS ALGORITHM IN HIGH DIMENSIONS

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Diffusion limits of MCMC methods in high dimensions provide a useful theoretical tool for studying computational complexity. In particular, they lead directly to precise estimates of the number of steps required to explore the target measure, in stationarity, as a function of the dimension of the state space. However, to date such results have mainly been proved for target measures with a product structure, severely limiting their applicability. The purpose of this paper is to study diffusion limits for a class of naturally occurring high-dimensional measures found from the approximation of measures on a Hilbert space which are absolutely continuous with respect to a Gaussian reference measure. The diffusion limit of a random walk Metropolis algorithm to an infinite-dimensional Hilbert space valued SDE (or SPDE) is proved, facilitating understanding of the computational complexity of the algorithm.

1. Introduction. Metropolis–Hastings methods [18, 21] form a widely used class of MCMC methods [19, 22] for sampling from complex probability distributions. It is, therefore, of considerable interest to develop mathematical analyses which explain the structure inherent in these algorithms, especially structure which is pertinent to understanding the computational complexity of the algorithm. Quantifying computational complexity of an MCMC method is most naturally undertaken by studying the behavior of the method on a family of probability distributions indexed by a parameter and studying the cost of the algorithm as a function of that parameter. In this paper we will study the cost as a function of dimension for algorithms applied to a family of probability distributions found from finite-dimensional approximation of a measure on an infinite-dimensional space.

Our interest is focused on Metropolis–Hastings MCMC methods [22]. We study the simplest of these, the random walk Metropolis algorithm (RWM). Let π be a target distribution on $\mathbb{R}^N$. To sample from π, the RWM algorithm creates a π-reversible Markov chain $\{x^n\}_{n=0}^\infty$ which moves from a current state x^0 to a new state x^1 via proposing a candidate y, using a symmetric Markov transition kernel such as a random walk, and accepting y with probability $\alpha(x^0, y)$, where

$\alpha(x, y) = 1 \wedge \frac{\pi(y)}{\pi(x)}$. Although the proposal is somewhat naive, within the class of all Metropolis–Hastings algorithms, the RWM is still used in many applications because of its simplicity. The only computational cost involved in calculating the acceptance probabilities is the relative ratio of densities $\frac{\pi(y)}{\pi(x)}$, as compared to, say, the Langevin algorithm (MALA) where one needs to evaluate the gradient of $\log \pi$.

A pioneering paper in the analysis of complexity for MCMC methods in high dimensions is [23]. This paper studied the behavior of random walk Metropolis methods when applied to target distributions with density

$$(1.1) \qquad \pi^N(x) = \prod_{i=1}^{N} f(x_i),$$

where $f(x)$ is a one-dimensional probability density function. The authors considered a proposal of the form

$$y = x + \sqrt{\delta}\rho,$$

$$\rho \overset{\mathcal{D}}{\sim} \mathrm{N}(0, \mathrm{I}_N),$$

and the objective was to study the complexity of the algorithm as a function of the dimension N of the state space. It was shown that choosing the proposal variance δ to scale as $\delta = 2\ell^2\lambda^2 N^{-1}$ with[3] $\lambda^{-2} = \int(\frac{f'}{f})^2 f \, dx$ ($\ell > 0$ is a parameter which we will discuss later) leads to an average acceptance probability of order 1 with respect to dimension N. Furthermore, with this choice of scaling, individual components of the resulting Markov chain converge to the solution of a stochastic differential equation (SDE). To state this, we define a continuous interpolant

$$(1.2) \qquad z^N(t) = (Nt - k)x^{k+1} + (k + 1 - Nt)x^k, \qquad k \leq Nt < k + 1.$$

Then [23] shows that, when the Markov chain is started in stationarity, $z^N \Rightarrow z$ as $N \to \infty$ in $C([0, T]; \mathbb{R})$ where z solves the SDE[4]

$$(1.3) \qquad \frac{dz}{dt} = \lambda^2 h(\ell)[\log f(z)]' + \sqrt{2\lambda^2 h(\ell)}\frac{dW}{dt},$$

$$(1.4) \qquad h(\ell) = 2\ell^2 \Phi\left(-\frac{\ell}{\sqrt{2}}\right).$$

Here Φ denotes the CDF of a standard normal distribution, "$\Rightarrow$" denotes weak convergence and $C([0, T], \mathbb{R})$ denotes the Banach space of real-valued continuous functions defined on the interval $[0, T]$ endowed with the usual supremum norm. Note that the invariant measure of the SDE (1.3) has the density f with respect

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[3]If f is the p.d.f. of a Gaussian on $\mathbb{R}$, then λ is its standard deviation.

[4]Our $h(\cdot)$ and ℓ are different from the h_{old} and ℓ_{old} used in [23]. However, they can be recovered from the identities $\ell_{\mathrm{old}}^2 = 2\lambda^2\ell^2$, $h_{\mathrm{old}}(\ell_{\mathrm{old}}) = 2\lambda^2 h(\ell)$.

to the Lebesgue measure. This weak convergence result leads to the interpretation that, started in stationarity and applied to target measures of the form (1.1), the RWM algorithm will take on the order of N steps to explore the invariant measure. Furthermore, it may be shown that the value of ℓ which maximizes $h(\ell)$ and, therefore, maximizes the speed of convergence of the limiting diffusion, leads to a universal acceptance probability, for random walk Metropolis algorithms applied to targets (1.1), of approximately 0.234.

These ideas have been generalized to other proposals, such as the MALA algorithm in [24]. For Langevin proposals, the scaling of δ which achieves order 1, acceptance probabilities is $\delta \propto N^{-1/3}$ and the choice of the constant of proportionality which maximizes the speed of the limiting SDE results from an acceptance probability of approximately 0.574. Note, in particular, that this method will take on the order of $N^{1/3}$ steps to explore the invariant distribution. This quantifies the advantage of using information about the gradient of $\log \pi$ in the proposal; RWM algorithms, which do not use this information, take on the order of N steps.

The work by Roberts and co-workers was among the first to develop a mathematical theory of Metropolis–Hastings methods in high dimension and does so in a fashion which leads to clear criteria which practitioners can use to optimize algorithmic performance, for instance, by tuning the acceptance probabilities to 0.234 (RWM) or 0.574 (MALA). Yet it is open to the criticism that, from a practitioner's perspective, target measures of the form (1.1) are too limited a class of probability distributions to be useful and, in any case, can be tackled by sampling a single one-dimensional target because of the product structure. There have been papers which generalize this work to target measures which retain the product structure inherent in (1.1), but are no longer i.i.d. (see [1, 25]),

$$(1.5) \qquad \pi_0^N(x) = \prod_{i=1}^{N} \lambda_i^{-1} f(\lambda_i^{-1} x_i).$$

However, the same criticism may be applied to this scenario as well.

Despite the apparent simplicity of target measures of the form (1.1) and (1.5), the intuition obtained from the study of Metropolis–Hastings methods applied to these models with product structure is, in fact, extremely valuable. The two key results which need to be transferred to a more general nonproduct measure setting are (i) the scaling of the proposal variance with N in order to ensure order one acceptance probabilities; (ii) the derivation of diffusion limits for the RWM algorithm with a time-scale factor which can be maximized over all acceptance probabilities. There is some work concerning scaling limits for MCMC methods applied to target measures which are not of product form; the paper [2] studies hierarchical target distributions; the paper [8] studies target measures which arise in nonlinear regression and have a mean field structure and the paper [9] studies target densities which are Gibbs measures. We add further to this literature on scaling limits for measures with nonproduct form by adopting the framework studied

in [4–6]. There the authors consider a target distribution π which lies in an infinite dimensional, real separable Hilbert space which is absolutely continuous with respect to a Gaussian measure π_0 with mean zero and covariance operator C (see Section 2.1 for details). The Radon–Nikodym derivative $\frac{d\pi}{d\pi_0}$ has the form

$$(1.6) \qquad \frac{d\pi}{d\pi_0} = M_\Psi \exp(-\Psi(x))$$

for a real valued π_0-measurable functional Ψ on the Hilbert space and M_Ψ a normalizing constant. In Section 3.1 we will specify and discuss the precise assumptions on Ψ which we adopt in this paper. This infinite-dimensional framework for the target measures, besides being able to capture a huge number of useful models arising in practice [16, 27], also has an inherent mathematical structure which makes it amenable to the derivation of diffusion limits in infinite dimensions, while retaining links to the product structure that has been widely studied. We highlight two aspects of this mathematical structure.

First, the theory of Gaussian measures naturally generalizes from $\mathbb{R}^N$ to infinite-dimensional Hilbert spaces. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle, \| \cdot \|)$ denote a real separable Hilbert space with full measure under μ_0 (Ψ will be densely defined on $\mathcal{H}$). The covariance operator $C : \mathcal{H} \mapsto \mathcal{H}$ is a self-adjoint, positive and trace class operator on $\mathcal{H}$ with a complete orthonormal eigenbasis $\{\lambda_j^2, \phi_j\}$,

$$C\phi_j = \lambda_j^2 \phi_j.$$

Henceforth, we assume that the eigenvalues are arranged in decreasing order and $\lambda_j > 0$. Any function $x \in \mathcal{H}$ can be represented in the orthonormal eigenbasis of C via the expansion

$$(1.7) \qquad x = \sum_{j=1}^{\infty} x_j \phi_j, \qquad x_j \overset{\text{def}}{=} \langle x, \phi_j \rangle.$$

Throughout this paper we will often identify the function x with its coordinates $\{x_j\}_{j=1}^{\infty} \in \ell^2$ in this eigenbasis, moving freely between the two representations. Note, in particular, that C is diagonal with respect to the coordinates in this eigenbasis. By the Karhunen–Loéve [13] expansion, a realization x from the Gaussian measure π_0 can be expressed by allowing the x_j to be independent random variables distributed as $x_j \sim N(0, \lambda_j^2)$. Thus, in the coordinates $\{x_j\}$, the prior has the product structure (1.5). For the random walk algorithm studied in this paper we assume that the eigenpairs $\{\lambda_j, \phi_j\}$ are known so that sampling from π_0 is straightforward.

The measure π is absolutely continuous with respect to π_0 and hence, any almost sure property under π_0 is also true under π. For example, it is a consequence of the law of large numbers that, almost surely with respect to π_0,

$$(1.8) \qquad \frac{1}{N} \sum_{j=1}^{N} \frac{x_j^2}{\lambda_j^2} \to 1 \qquad \text{as } N \to \infty.$$

This also holds almost surely with respect to π, implying that a typical draw from the target measure π must behave like a typical draw from π_0 in the large j co-ordinates.[5] This offers hope that ideas from the product case are applicable to measures π given by (1.6) as well. However, the presence of Ψ prevents use of the techniques from previous work on this problem; the fact that individual components of the Markov chain converge to a scalar SDE, as proved in [23], is a direct consequence of the product structure inherent in (1.1) or (1.5). For target measures of the form (1.6), this structure is not present and individual components of the Markov chain cannot be expected to converge to a scalar SDE. However, it is natural to expect convergence of the entire Markov chain to an infinite-dimensional continuous time stochastic process and the purpose of this paper is to carry out such a program.

Thus, the second fact which makes the target measure (1.6) attractive from the point of view of establishing diffusion limits is that fact that, as proved in a series of recent papers [15, 17], it is invariant for Hilbert-space valued SDEs (or stochastic PDES–SPDEs) with the form

$$(1.9) \qquad \frac{dz}{dt} = -h(\ell)\big(z + C\nabla\Psi(z)\big) + \sqrt{2h(\ell)}\,\frac{dW}{dt}, \qquad z(0) = z^0,$$

where W is a Brownian motion (see [13]) in $\mathcal{H}$ with covariance operator C. Thus, the above result from SPDE theory gives us a natural candidate for the infinite-dimensional limit of an MCMC method. We will prove such a limit for a RWM algorithm with proposal covariance $\frac{2\ell^2}{N}C$. Moreover, we will show that the time constant $h(\ell)$ is maximized for an average acceptance probability of 0.234, as obtained in [23] in the product case.

These measures π given by (1.6) have a number of features which will enable us to develop the ideas of diffusion limits for MCMC methods as originally introduced in the i.i.d. product case. Carrying out this program is worthwhile because measures of the form given by (1.6) arise naturally in a range of applications. In particular, they arise in the context of nonparametric regression in Bayesian statistics where the parameter space is an infinite-dimensional function space. The measure π_0 is the prior and Ψ the log likelihood function. Such Bayesian inverse problems are overviewed in [27]. Another class of problems leading to measures of the form (1.6) are conditioned diffusions (see [16]).

To sample from π numerically we need a finite-dimensional target measure. To this end, let $\Psi^N(\cdot) = \Psi(P^N\cdot)$ where P^N denotes projection[6] (in $\mathcal{H}$) onto the first

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[5]For example, if μ_0 is the Gaussian measure associated with Brownian motion on a finite interval, then (1.8) is an expression for the variance scale in the quadratic variation, and this is preserved under changes of measure such as the Girsanov formula.

[6]Actually Ψ is only densely defined on $\mathcal{H}$ but the projection P^N can also be defined on this dense subset.

N eigenfunctions of C. Then consider the target measure π^N with the form

$$(1.10) \qquad \frac{d\pi^N}{d\pi_0}(x) \propto \exp(-\Psi^N(x)).$$

This measure can be factored as the product of two independent measures: it coincides with π_0 on $\mathcal{H} \setminus P^N \mathcal{H}$ and has a density with respect to Lebesgue measure on $P^N \mathcal{H}$, in the coordinates $\{x_j\}_{j=1}^N$. In computational practice we implement a random walk method on $\mathbb{R}^N$ in the coordinate system $\{x_j\}_{j=1}^N$, enabling us to sample from π^N in $P^N \mathcal{H}$. However, in order to facilitate a clean analysis, it is beneficial to write this finite-dimensional random walk method in $\mathcal{H}$, noting that the coordinates $\{x_j\}_{j=N+1}^\infty$ in the representation of functions sampled from π^N do not then change. We consider proposal distributions for the RWM which exploit the covariance structure of π_0 and can be expressed in $\mathcal{H}$ as

$$(1.11) \quad y = x + \sqrt{\frac{2\ell^2}{N}} C^{1/2} \xi \qquad \text{where } \xi = \sum_{j=1}^N \xi_j \phi_j \qquad \text{with } \xi_j \overset{\mathcal{D}}{\sim} N(0,1) \text{ i.i.d.}$$

Note that our proposal variance scales as $N^{-\gamma}$ with $\gamma = 1$. The choice of γ in the proposal variance affects the scale of the proposal moves and identifying the optimal choice for γ is a delicate exercise. The larger γ is, the more "localized" the proposed move is and, therefore, for the algorithm to explore the state space rapidly, γ needs to be as small as possible. However, if we take γ arbitrarily small, then the acceptance probability decreases to zero very rapidly as a function of N. In fact, it was shown in [4–6] that, for a variety of Metropolis–Hastings proposals, there is $\gamma_c > 0$ such that choice of $\gamma < \gamma_c$ leads to average acceptance probabilities which are smaller than any inverse power of N. Thus, in higher dimensions, smaller values of γ lead to very poor mixing because of the negligible acceptance probability. However, it turns out that at the critical value γ_c, the acceptance probability is $\mathcal{O}(1)$ as a function of N. In [4, 6], the value of γ_c was identified to be 1 and 1/3 for the RWM and MALA, respectively. Finally, when using the scalings leading to $\mathcal{O}(1)$ acceptance probabilities, it was also shown that the mean square distance moved is maximized by choosing the acceptance probabilities to be 0.234 or 0.574 as in the i.i.d. product case (1.1). Guided by this intuition, we have chosen $\gamma = \gamma_c = 1$ for our RWM proposal variance which, as we will prove below, leads to $\mathcal{O}(1)$ acceptance probabilities.

Summarizing the discussion so far, our goal is to obtain an invariance principle for the RWM Markov chain with proposal (1.11) when applied to target measures of the form (1.6). The diffusion limit will be obtained in stationarity and will be given by the SPDE (1.9). We show that the continuous time interpolant z^N of the Markov chain $\{x^k\}$ defined by (1.2) converges to z solving (1.9). This will show that, in stationarity and properly scaled to achieve $\mathcal{O}(1)$ acceptance probabilities, the random walk Metropolis algorithm takes $\mathcal{O}(N)$ steps to explore the target distribution. From a practical point of view, the take home message of this work is

that standard RWM algorithms applied to approximations of target measures with the form (1.6) can be tuned to behave optimally by adjusting the acceptance probability to be approximately 0.234 in the case where the proposal covariance is proportional to the covariance C in the reference measure. This will lead to $\mathcal{O}(N)$ steps to explore the target measure in stationarity. This extends the work in [23] and shows that the ideas developed there apply to nontrivial high-dimensional targets arising in applications. Although we only analyze the RWM proposal (1.11), we believe that our techniques can be applied to a larger class of Metropolis–Hastings methods, including the MALA algorithm, and/or RWM methods with isotropic proposal variance. In this latter case we expect to get a different (nonpreconditioned) π-invariant SPDE as the limit when the dimension goes to infinity (see [15, 17] for analysis of these SPDEs) and a different (more severe) restriction on the scaling of the proposal variance with N; however, we conjecture that the optimal acceptance probability would not be changed. The proposal that we study in this paper relies on knowledge of the eigenstructure of the covariance operator of the prior or reference measures π_0. In some applications, this may be a reasonable assumption, for example, for conditioned diffusions or for PDE inverse problems in simple geometries. For others it may not, and then the isotropic proposal covariance is more natural.

We analyze the RWM algorithm started at stationarity, and thus do not attempt to answer the question of "burn-in time": the number of steps required to reach stationarity and how the proposal scaling affects the rate of convergence. These are important questions which we hope to answer in a future paper. Furthermore, practitioners wishing to sample from probability measures on function space with the form (1.6) should be aware that for some examples, new generalizations of random walk Metropolis algorithms, defined on function space, can be more efficient than the standard random walk methods analyzed in this paper [5, 6]; whether or not they are more efficient depends on a trade-off between number of steps to explore the measure (which is lower for the new generalized methods) and cost per step (which can be higher, but may not be).

There exist several methods in the literature to prove invariance principles. For instance, because of the reversibility of the RWM Markov chain, utilizing the abstract but powerful theory of Dirichlet forms [20] is appealing. Another alternative is to show the convergence of generators of the associated Markov processes [14] as used in [23]. However, we chose a more "hands on" approach using simple probabilistic tools, thus gaining more intuition about the RWM algorithm in higher dimensions. We show that with the correct choice of scaling, the one step transition for the RWM Markov chain behaves nearly like an Euler scheme applied to (1.9). Since the noise enters (1.9) additively, the induced Itô map which takes Wiener trajectories into solutions is continuous in the supremum-in-time topology. This fact, which would not be true if (1.9) had multiplicative noise, allows to employ an argument simpler than the more general techniques often used (see [14]).

We first show that the martingale increments converge weakly to a Hilbert space-valued Wiener process using a martingale central limit theorem [3]. Since weak convergence is preserved under a continuous map, the fact that the Itô map is continuous implies the RWM Markov chain converges to the SPDE (1.9). Finally, we emphasize that diffusion limits for the RWM proposal are necessarily of weak convergence type. However, strong convergence results are available for the MALA algorithm, in fixed finite dimension (see [7]).

1.1. *Organization of the paper.* We start by setting up the notation that is used for the remainder of the paper in Section 2. We then investigate the mathematical structure of the RWM algorithm when applied to target measures of the form (1.10). Before presenting details, a heuristic but detailed outline of the proof strategy is given for communicating the main ideas. In Section 3 we state our assumptions and give the proof of the main theorem at a high level, postponing proofs of some technical estimates. In Section 4 we prove the invariance principle for the noise process. Section 5 contains the proof of the drift and diffusion estimates. All universal constants, unless otherwise stated, are denoted by the letter M whose precise value might vary from one line to the next.

2. Diffusion limits of the RWM algorithm.

In this section we state the main theorem, set it in context and explain the proof technique. We first introduce an approximation of the measure π, namely π^N, which is finite dimensional. We then state the main theorem concerning a diffusion limit of the algorithm and sketch the ideas of the proof so that technical details in later sections can be readily digested.

2.1. *Preliminaries.* Recall that $\mathcal{H}$ is a separable Hilbert space of real-valued functions with inner-product and norm $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a positive, trace class operator on $\mathcal{H}$. Let $\{\phi_j, \lambda_j^2\}$ be the eigenfunctions and eigenvalues of C, respectively, so that

$$C\phi_j = \lambda_j^2 \phi_j, \qquad j \in \mathbb{N}.$$

We assume a normalization under which $\{\phi_j\}$ forms a complete orthonormal basis in $\mathcal{H}$. We also assume that the eigenvalues are arranged in decreasing order. For every $x \in \mathcal{H}$ we have the representation (1.7). Using this expansion, we define the Sobolev spaces $\mathcal{H}^r, r \in \mathbb{R}$, with the inner-products and norms defined by

$$(2.1) \qquad \langle x, y \rangle_r \overset{\text{def}}{=} \sum_{j=1}^{\infty} j^{2r} x_j y_j, \qquad \|x\|_r^2 \overset{\text{def}}{=} \sum_{j=1}^{\infty} j^{2r} x_j^2.$$

Notice that $\mathcal{H}^0 = \mathcal{H}$. Furthermore, $\mathcal{H}^r \subset \mathcal{H} \subset \mathcal{H}^{-r}$ for any $r > 0$. For $r \in \mathbb{R}$, let $B_r : \mathcal{H} \mapsto \mathcal{H}$ denote the operator which is diagonal in the basis $\{\phi_j\}$ with diagonal entries j^{2r}, that is,

$$B_r \phi_j = j^{2r} \phi_j$$

so that $B_r^{1/2}\phi_j = j^r\phi_j$. The operator B_r lets us alternate between the Hilbert space $\mathcal{H}$ and the Sobolev spaces $\mathcal{H}^r$ via the identities

$$(2.2) \qquad \langle x, y\rangle_r = \langle B_r^{1/2}x, B_r^{1/2}y\rangle, \qquad \|x\|_r^2 = \|B_r^{1/2}x\|^2.$$

Let $\otimes$ denote the outer product operator in $\mathcal{H}$ defined by

$$(2.3) \qquad (x \otimes y)z \stackrel{\text{def}}{=} \langle y, z\rangle x \qquad \forall x, y, z \in \mathcal{H}.$$

For an operator $L: \mathcal{H}^r \mapsto \mathcal{H}^l$, we denote the operator norm on $\mathcal{H}$ by $\|\cdot\|_{\mathcal{L}(\mathcal{H}^r, \mathcal{H}^l)}$ defined by

$$\|L\|_{\mathcal{L}(\mathcal{H}^r, \mathcal{H}^l)} \stackrel{\text{def}}{=} \sup_{\|x\|_r=1} \|Lx\|_l.$$

For self-adjoint L and $r = l = 0$ this is, of course, the spectral radius of L. For a positive, self-adjoint operator $D: \mathcal{H} \mapsto \mathcal{H}$, define its trace as

$$\text{trace}(D) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \langle \phi_j, D\phi_j\rangle.$$

Since $\text{trace}(D)$ does not depend on the orthonormal basis, an operator D is said to be trace class if $\text{trace}(D) < \infty$ for some, and hence any, orthonormal basis $\{\phi_j\}$.

Let π_0 denote a mean zero Gaussian measure on $\mathcal{H}$ with covariance operator C, that is, $\pi_0 \stackrel{\text{def}}{=} N(0, C)$. If $x \stackrel{\mathcal{D}}{\sim} \pi_0$, then the x_j in (1.7) are independent $N(0, \lambda_j^2)$ Gaussians and we may write (Karhunen–Loéve)

$$(2.4) \qquad x = \sum_{j=1}^{\infty} \lambda_j \rho_j \phi_j \qquad \text{with } \rho_j \stackrel{\mathcal{D}}{\sim} N(0, 1) \text{ i.i.d.}$$

Since $\|B^{-1/2}\phi_k\|_r = \|\phi_k\| = 1$, we deduce that $\{B_r^{-1/2}\phi_k\}$ form an orthonormal basis for $\mathcal{H}^r$ and, therefore, we may write (2.4) as

$$(2.5) \qquad x = \sum_{j=1}^{\infty} \lambda_j j^r \rho_j B_r^{-1/2}\phi_j \qquad \text{with } \rho_j \stackrel{\mathcal{D}}{\sim} N(0, 1) \text{ i.i.d.}$$

If Ω denotes the probability space for sequences $\{\rho_j\}_{j\geq 1}$, then the sum converges in $L^2(\Omega; \mathcal{H}^r)$ as long as $\sum_{j=1}^{\infty} \lambda_j^2 j^{2r} < \infty$. Thus, under this condition, the distribution induced by π_0 may be viewed as that of a centered Gaussian measure on $\mathcal{H}^r$ with covariance operator C_r given by

$$(2.6) \qquad C_r = B_r^{1/2} C B_r^{1/2}.$$

The assumption on summability is the usual trace-class condition for Gaussian measures on a Hilbert space: $\text{trace}(C_r) < \infty$. In what follows, we freely alternate between the Gaussian measures $N(0, C)$ on $\mathcal{H}$ and $N(0, C_r)$ on $\mathcal{H}^r$, for values of r for which the trace-class property of C_r holds.

Our goal is to sample from a measure π on $\mathcal{H}$ given by (1.6),

$$\frac{d\pi}{d\pi_0} = M_\Psi \exp(-\Psi(x))$$

with π_0 as constructed above. Frequently in applications, the functional Ψ may not be defined on all of $\mathcal{H}$, but only on a subset $\mathcal{H}^r \subset \mathcal{H}$ for some exponent $r > 0$. For instance, if $\mathcal{H} = L_2([0, 1])$, the functional Ψ might only act on continuous functions, in which case it is natural to define Ψ on some Sobolev space $\mathcal{H}^r[0, 1]$ for $r > \frac{1}{2}$. Even though the Gaussian measure π_0 is defined on $\mathcal{H}$, depending on the decay of the eigenvalues of C, there exists an entire range of values r such that trace$(C_r) < \infty$ so that the measure π_0 has full support on $\mathcal{H}^r$, that is, $\pi_0(\mathcal{H}^r) = 1$. From now onward we fix a distinguished exponent $s \geq 0$ and assume that $\Psi : \mathcal{H}^s \mapsto \mathbb{R}$ and that the prior is chosen so that trace$(C_s) < \infty$. Then $\pi_0 \sim N(0, C)$ on $\mathcal{H}$ and $\pi(\mathcal{H}^s) = 1$; in addition, we may view π_0 as a Gaussian measure $N(0, C_s)$ on $\mathcal{H}^s$. The precise connection between the exponent s and the eigenvalues of C is given in Section 3.1.

In order to sample from π we first approximate it by a finite-dimensional measure. Recall that

$$(2.7) \qquad \widehat{\phi}_k \overset{\text{def}}{=} B_s^{-1/2} \phi_k$$

form an orthonormal basis for $\mathcal{H}^s$. For $N \in \mathbb{N}$, let $P^N : \mathcal{H}^s \mapsto X^N \subset \mathcal{H}^s$ be the projection operator in $\mathcal{H}^s$ onto $X^N \overset{\text{def}}{=} \text{span}\{\widehat{\phi}_1, \widehat{\phi}_2, \ldots, \widehat{\phi}_N\}$, that is,

$$P^N x \overset{\text{def}}{=} \sum_{j=1}^{N} x_j \widehat{\phi}_j \qquad \text{where } x_j = \langle x, \widehat{\phi}_j \rangle_s, x \in \mathcal{H}^s.$$

This shows that X^N is isomorphic to $\mathbb{R}^N$. Next, we approximate Ψ by $\Psi^N : X^N \mapsto \mathbb{R}$ and attempt to sample from the following approximation to π, namely,

$$\frac{d\pi^N}{d\pi_0}(x) \overset{\text{def}}{=} M_{\Psi^N} \exp(-\Psi^N(x)) \qquad \text{where } \Psi^N(x) \overset{\text{def}}{=} \Psi(P^N x).$$

Note that $\nabla \Psi^N(x) = P^N \nabla \Psi(P^N x)$ and $\partial^2 \Psi^N(x) = P^N \partial^2 \Psi(P^N x) P^N$. The constant M_{Ψ^N} is chosen so that $\pi^N(\mathcal{H}^s) = 1$. It may be shown that, for large N, the measure π^N is close to the measure π in the Hellinger metric (see [12]). Set

$$(2.8) \qquad C^N \overset{\text{def}}{=} P^N C P^N, \qquad C_r^N \overset{\text{def}}{=} B_r^{1/2} C^N B_r^{1/2}.$$

Notice that on X^N, π^N has Lebesgue density[7]

$$(2.9) \qquad \begin{aligned} \pi^N(x) &= M_{\Psi^N} \exp\left(-\Psi^N(x) - \tfrac{1}{2}\langle P^N x, C^{-1}(P^N x) \rangle\right), \qquad x \in X^N \\ &= M_{\Psi^N} \exp\left(-\Psi^N(x) - \tfrac{1}{2}\langle x, (C^N)^{-1} x \rangle\right) \end{aligned}$$

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[7]For ease of notation we do not distinguish between a measure and its density.

since C^N is invertible on X^N because the eigenvalues are assumed to be strictly positive. On $\mathcal{H}^s \setminus X^N$ we have that $\pi^N = \pi_0$. Later we will impose natural assumptions on Ψ (and hence, on Ψ^N) which are motivated by applications.

2.2. *The algorithm.* Our goal is now to sample from (2.9) with $x \in X^N$. As explained in the Introduction, we use a RWM proposal with covariance operator $2\frac{\ell^2}{N}C$ on $\mathcal{H}$ given by (1.11). The noise ξ is finite dimensional and is independent of x. Hence, even though the Markov chain evolves in $\mathcal{H}^s$, x and y in (1.11) differ only in the first N coordinates when written in the eigenbasis of C; as a consequence, the Markov chain does not move at all in $\mathcal{H}^s \setminus P^N\mathcal{H}^s$ and can be implemented in $\mathbb{R}^N$. However the analysis is cleaner when written in $\mathcal{H}^s$. The acceptance probability also only depends on the first N coordinates of x and y and has the form

$$(2.10) \qquad \alpha(x, \xi) = 1 \wedge \exp(Q(x, \xi)),$$

where

$$(2.11) \qquad \begin{aligned} Q(x, \xi) &\stackrel{\text{def}}{=} \tfrac{1}{2}\|C^{-1/2}(P^N x)\|^2 - \tfrac{1}{2}\|C^{-1/2}(P^N y)\|^2 \\ &\quad + \Psi^N(x) - \Psi^N(y). \end{aligned}$$

The Markov chain for $\{x^k\}$, $k \geq 0$ is then given by

$$(2.12) \quad x^{k+1} = \gamma^{k+1}y^{k+1} + (1 - \gamma^{k+1})x^k \quad \text{and} \quad y^{k+1} = x^k + \sqrt{\frac{2\ell^2}{N}}C^{1/2}\xi^{k+1}$$

with

$$\gamma^{k+1} \stackrel{\text{def}}{=} \gamma(x^k, \xi^{k+1}) \stackrel{\mathcal{D}}{\sim} \text{Bernoulli}(\alpha(x^k, \xi^{k+1})) \quad \text{and} \quad \xi^{k+1} = \sum_{i=1}^{N}\xi_i^{k+1}\phi_i$$

$$\text{where } \xi_i^{k+1} \stackrel{\mathcal{D}}{\sim} \text{N}(0, 1) \text{ i.i.d.}$$

with some initial condition x^0. The random variables ξ^k and x^0 are independent of one another. Furthermore, conditional on $\alpha(x^{k-1}, \xi^k)$, the Bernoulli random variables γ^k are chosen independently of all other sources of randomness. This can be seen in the usual way by introducing an i.i.d. sequence of uniform random variables Unif$[0, 1]$ and using these for each k to construct the Bernoulli random variable.

In summary, the Markov chain that we have described in $\mathcal{H}^s$ is, when projected into coordinates $\{x_j\}_{j=1}^N$, equivalent to a standard random walk Metropolis method for the Lebesgue density (2.9) with proposal variance given by C^N on $\mathcal{H}$. Recall that the target measure π in (1.6) is the invariant measure of the SPDE (1.9). Our goal is to obtain an invariance principle for the continuous interpolant (1.2) of the

Markov chain $\{x^k\}$ started in stationarity: to show weak convergence of $z^N(t)$ to the solution $z(t)$ of the SPDE (1.9), as the dimension $N \to \infty$.

In the rest of the section, we will give a heuristic outline of our main argument. The emphasis will be on the proof strategy and main ideas. So we will not yet prove the error bounds and use the symbol "$\approx$" to indicate so. Once the main skeleton is outlined, we retrace our arguments and make them rigorous in Sections 3, 4 and 5.

2.3. *Main theorem and implications.* As mentioned earlier for fixed N, the Markov chain evolves in $X^N \subset \mathcal{H}^s$ and we prove the invariance principle for the Markov chain in the Hilbert space $\mathcal{H}^s$ as N goes to infinity. Define the constant β,

$$(2.13) \qquad \beta \overset{\text{def}}{=} 2\Phi(-\ell/\sqrt{2}),$$

where Φ denotes the CDF of the standard normal distribution. Note that with this definition of β, the time scale $h(\ell)$ appearing in (1.9), and defined in (1.4), is given by $h(\ell) = \ell^2 \beta$. The following is the main result of this article (it is stated precisely, with conditions, as Theorem 3.6):

MAIN THEOREM. *Let the initial condition x^0 of the RWM algorithm be such that $x^0 \overset{\mathcal{D}}{\sim} \pi^N$ and let $z^N(t)$ be a piecewise linear, continuous interpolant of the RWM algorithm (2.12) as defined in (1.2). Then $z^N(t)$ converges weakly in $C([0, T], \mathcal{H}^s)$ to the diffusion process $z(t)$ given by (1.9) with $z(0) \overset{\mathcal{D}}{\sim} \pi$.*

We will now explain the following two important implications of this result:

- it demonstrates that, in stationarity, the work required to explore the invariant measure scales as $O(N)$;
- it demonstrates that the speed at which the invariant measure is explored, again in stationarity, is maximized by tuning the average acceptance probability to 0.234.

The first implication follows from (1.2) since this shows that $O(N)$ steps of the Markov chain (2.12) are required for $z^N(t)$ to approximate $z(t)$ on a time interval $[0, T]$ long enough for $z(t)$ to have explored its invariant measure. The second implication follows from (1.9) for $z(t)$ itself. The maximum of the time-scale $h(\ell)$ over the parameter ℓ (see [23]) occurs at a universal acceptance probability of $\widehat{\beta} = 0.234$, to three decimal places. Thus, remarkably, the optimal acceptance probability identified in [23] for product measures, is also optimal for the nonproduct measures studied in this paper.

2.4. *Proof strategy.* Let $\mathcal{F}_k$ denote the sigma algebra generated by $\{x^n, \xi^n, \gamma^k, n \le k\}$. We denote the conditional expectations $\mathbb{E}(\cdot|\mathcal{F}_k)$ by $\mathbb{E}_k(\cdot)$. We first compute the one-step expected drift of the Markov chain $\{x^k\}$. For notational convenience let $x^0 = x$ and $\xi^1 = \xi$. We set $\xi^0 = 0$ and $\gamma^0 = 0$. Then, under the assumptions

on Ψ, Ψ^N given in Section 3.1, we prove the following proposition estimating the mean one-step drift and diffusion. The proof is given in Sections 5.2 and 5.3.

PROPOSITION 2.1.   *Let Assumptions* 3.1 *and* 3.4 *(below) hold. Let* $\{x^k\}$ *be the RWM Markov chain with* $x^0 = x \overset{\mathcal{D}}{\sim} \pi^N$. *Then*

$$(2.14) \qquad N\mathbb{E}_0(x^1 - x) = -\ell^2\beta(P^N x + C^N \nabla \Psi^N(x)) + r^N,$$

$$(2.15) \quad N\mathbb{E}_0[(x^1 - x) \otimes (x^1 - x)] = 2\ell^2\beta C^N + E^N,$$

where the error terms r^N *and* E^N *satisfy* $\mathbb{E}^{\pi^N}\|r^N\|_s^2 \to 0$, $\mathbb{E}^{\pi^N} \sum_{i=1}^N |\langle \phi_i, E^N \phi_i \rangle_s| \to 0$ *and* $\mathbb{E}^{\pi^N}|\langle \phi_i, E^N \phi_j \rangle_s| \to 0$ *as* $N \to \infty$, *for any pair of indices* i, j *and for* s *appearing in Assumptions* 3.1.

Thus the discrete time Markov chain $\{x^k\}$ obtained by the successive accepted samples of the RWM algorithm has approximately the expected drift and covariance structure of the SPDE (1.9). It is also crucial to our subsequent argument involving the martingale central limit theorem that the error terms r^N and E^N converge to zero in the Hilbert space $\mathcal{H}^s$ norm and inner-product as stated.

With this in hand, we need to establish the appropriate invariance principle to show that the dynamics of the Markov chain $\{x^k\}$, when seen as the values of a continuous time process on a time mesh with steps of $O(1/N)$, converges weakly to the law of the SPDE given in (1.9) on $C([0, T], \mathcal{H}^s)$. To this end we define, for $k \geq 0$,

$$(2.16) \qquad \begin{aligned} m^N(\cdot) &\overset{\text{def}}{=} P^N(\cdot) + C^N \nabla \Psi^N(\cdot) \Gamma^{k+1,N} \\ &\overset{\text{def}}{=} \sqrt{\frac{N}{2\ell^2\beta}}(x^{k+1} - x^k - \mathbb{E}_k(x^{k+1} - x^k)), \end{aligned}$$

$$(2.17) \qquad r^{k+1,N} \overset{\text{def}}{=} N\mathbb{E}_k(x^{k+1} - x^k) + \ell^2\beta(P^N x^k + C^N \nabla \Psi^N(x^k)),$$

$$(2.18) \qquad E^{k+1,N} \overset{\text{def}}{=} N\mathbb{E}_k[(x^{k+1} - x^k) \otimes (x^{k+1} - x^k)] - 2\ell^2\beta C^N$$

with $E^{0,N}, \Gamma^{0,N}, r^{0,N} = 0$. Notice that for fixed N, $\{r^{k,N}\}_{k\geq 1}, \{E^{k,N}\}_{k\geq 1}$ are, since $x^0 \sim \pi^N$, stationary sequences.

By definition,

$$(2.19) \qquad x^{k+1} = x^k + \mathbb{E}_k(x^{k+1} - x^k) + \sqrt{\frac{2\ell^2\beta}{N}}\Gamma^{k+1,N}.$$

From (2.14) in Proposition 2.1, for large enough N,

$$(2.20) \qquad \begin{aligned} x^{k+1} &\approx x^k - \frac{\ell^2\beta}{N}(P^N x^k + C^N \nabla \Psi^N(x^k)) + \sqrt{\frac{2\ell^2\beta}{N}}\Gamma^{k+1,N} \\ &= x^k - \frac{\ell^2\beta}{N}m^N(x^k) + \sqrt{\frac{2\ell^2\beta}{N}}\Gamma^{k+1,N}. \end{aligned}$$

From the definition of $\Gamma^{k,N}$ in (2.16), and from (2.15) in Proposition 2.1,

$$\mathbb{E}_k(\Gamma^{k+1,N}) = 0 \quad \text{and} \quad \mathbb{E}_k(\Gamma^{k+1,N} \otimes \Gamma^{k+1,N}) \approx C^N.$$

Therefore, for large enough N, equation (2.20) "resembles" the Euler scheme for simulating the finite-dimensional approximation of the SPDE (1.9) on $\mathbb{R}^N$, with drift function $m^N(\cdot)$ and covariance operator C^N:

$$x^{k+1} \approx x^k - h(\ell)m^N(x^k)\Delta t + \sqrt{2h(\ell)\Delta t}\,\Gamma^{k+1,N} \qquad \text{where } \Delta t \stackrel{\text{def}}{=} \frac{1}{N}.$$

This is the key idea underlying our main result (Theorem 3.6): the Markov chain (2.12) looks like a weak Euler approximation of (1.9).

Note that there is an important difference in analyzing the weak convergence from the traditional Euler scheme. In our case, for any fixed $N \in \mathbb{N}$, $\Gamma^{k,N} \in X^N$ is finite dimensional, but clearly the dimension of $\Gamma^{k,N}$ grows with N. Also, the distribution of the initial condition $x(0) \stackrel{\mathcal{D}}{\sim} \pi^N$ changes with N, unlike the case of the traditional Euler scheme where the distribution of $x(0)$ does not change with N. Moreover, for any fixed N, the "noise" process $\{\Gamma^{k,N}\}$ are not formed of independent random variables. However, they are identically distributed (a stationary sequence) because the Metropolis algorithm preserves stationarity. To obtain an invariance principle, we first use a version of the martingale central limit theorem (Proposition 4.1) to show that the noise process $\{\Gamma^{k,N}\}$, when rescaled and summed, converges weakly to a Brownian motion on $C([0,T], \mathcal{H}^s)$ with covariance operator C_s, for any $T = \mathcal{O}(1)$. We then use continuity of an appropriate Itô map to deduce the desired result.

Before we proceed, we introduce some notation. Fix $T > 0$, and define

$$(2.21) \qquad \Delta t \stackrel{\text{def}}{=} 1/N, \qquad t^k \stackrel{\text{def}}{=} k\Delta t, \qquad \eta^{k,N} \stackrel{\text{def}}{=} \sqrt{\Delta t}\sum_{l=1}^{k}\Gamma^{l,N}$$

and

$$(2.22) \qquad W^N(t) \stackrel{\text{def}}{=} \eta^{\lfloor Nt \rfloor, N} + \frac{Nt - \lfloor Nt \rfloor}{\sqrt{N}}\Gamma^{\lfloor Nt \rfloor + 1, N}, \qquad t \in [0, T].$$

Let $W(t), t \in [0, T]$ be an $\mathcal{H}^s$ valued Brownian motion with covariance operator C_s. Using a martingale central limit theorem, we will prove the following proposition in Section 4.

PROPOSITION 2.2. *Let Assumptions 3.1 (below) hold. Let $x^0 \sim \pi^N$. The process $W^N(t)$ defined in (2.22) converges weakly to W in $C([0,T], \mathcal{H}^s)$ as N tends to ∞, where W is a Brownian motion in time with covariance operator C_s in $\mathcal{H}^s$ and s is defined in Assumptions 3.1. Furthermore, the pair $(x^0, W^N(t))$ converges weakly to (z^0, W) where $z^0 \sim \pi$ and Brownian motion W is independent of the initial condition z^0 almost surely.*

Using this invariance principle for the noise process and the fact that the noise process is additive (the diffusion coefficient is constant), the invariance principle for the Markov chain follows from a continuous mapping argument which we now outline. For any $(z^0, W) \in \mathcal{H}^s \times C([0, T]; \mathcal{H}^s)$, we define the Itô map $\Theta : \mathcal{H}^s \times C([0, T]; \mathcal{H}^s) \to C([0, T]; \mathcal{H}^s)$ by $\Theta : (z^0, W) \mapsto z$ where z solves

$$(2.23) \qquad z(t) = z^0 - h(\ell) \int_0^t (z(s) + C \nabla \Psi(z(s))) \, ds + \sqrt{2h(\ell)} W(t)$$

for all $t \in [0, T]$ and $h(\ell) = \ell^2 \beta$ is as defined in (1.4). Thus $z = \Theta(z^0, W)$ solves the SPDE (1.9) with $h(\ell) = \ell^2 \beta$. We will see in Lemma 3.7 that Θ is a continuous map from $\mathcal{H}^s \times C([0, T]; \mathcal{H}^s)$ into $C([0, T]; \mathcal{H}^s)$.

We now define the piecewise constant interpolant of x^k,

$$(2.24) \qquad \bar{z}^N(t) = x^k \qquad \text{for } t \in [t^k, t^{k+1}).$$

Set

$$(2.25) \qquad d^N(x) \stackrel{\text{def}}{=} N \mathbb{E}_0(x^1 - x).$$

Note that $d_N(x) \approx -h(\ell) m_N(x)$. We can use $\bar{z}^N$ to construct a continuous piecewise linear interpolant of x^k by defining

$$(2.26) \qquad z^N(t) = z^0 + \int_0^t d^N(\bar{z}^N(s)) \, ds + \sqrt{2h(\ell)} W^N(t).$$

Notice that $d^N(x)$ defined in (2.25) is a function which depends on arbitrary $x = x^0$ and averages out the randomness in x^1 conditional on fixing $x = x^0$. We may then evaluate this function at any $x \in \mathcal{H}^s$ and, in particular, at $\bar{z}^N(s)$ as in (2.26). Use of the stationarity of the sequence x^k, together with equations (2.19), (2.21) and (2.22), reveals that the definition (2.26) coincides with that given in (1.2). Using the closeness of d^N and $-h(\ell) m^N$, of z^N and $\bar{z}^N$ and of m^N and the desired limiting drift, we will see that there exists a $\widehat{W}^N \Rightarrow W$ as $N \to \infty$, such that

$$(2.27) \qquad z^N(t) = z^0 - h(\ell) \int_0^t (z^N(s) + C \nabla \Psi(z^N(s))) \, ds + \sqrt{2h(\ell)} \widehat{W}^N(t),$$

so that $z^N = \Theta(z^0, \widehat{W}^N)$. By the continuity of Θ we will show, using the continuous mapping theorem, that

$$(2.28) \qquad z^N = \Theta(z^0, \widehat{W}^N) \quad \implies \quad z = \Theta(z^0, W) \qquad \text{as } N \to \infty.$$

It will be important to show that the weak limit of $(z^0, \widehat{W}^N)$, namely (z^0, W), comprises of two independent random variables z^0 (from the stationary distribution) and W.

The weak convergence in (2.28) is the principal result of this article and is stated precisely in Theorem 3.6. To summarize, we have argued that the RWM is well approximated by an Euler approximation of (1.9). The Euler approximation itself can be seen as an approximate solution of (1.9) with a modified Brownian motion. As $N \to \infty$, all approximation errors go to zero in the appropriate sense and one deduces that the RWM algorithm converges to the solution of (1.9).

2.5. *A framework for expected drift and diffusion.* We now turn to the question of how the RWM algorithm produces the appropriate drift and covariance encapsulated in Proposition 2.1. This result, which shows that the algorithm (approximately) performs a noisy steepest ascent process, is at the heart of why the Metropolis algorithm works. In the rest of this section we set up a framework which will be used for deriving the expected drift and diffusion terms.

Recall the setup from Section 2. Starting from (2.11), after some algebra we obtain

$$(2.29) \qquad Q(x, \xi) = -\sqrt{\frac{2\ell^2}{N}} \langle \zeta, \xi \rangle - \frac{\ell^2}{N} \|\xi\|^2 - r(x, \xi),$$

where we have defined

$$(2.30) \qquad \zeta \stackrel{\text{def}}{=} C^{-1/2}(P^N x) + C^{1/2} \nabla \Psi^N(x),$$

$$(2.31) \qquad r(x, \xi) \stackrel{\text{def}}{=} \Psi^N(y) - \Psi^N(x) - \langle \nabla \Psi^N(x), P^N y - P^N x \rangle.$$

REMARK 2.3. If $x \stackrel{\mathcal{D}}{\sim} \pi_0$ in $\mathcal{H}^s$, then the random variable $C^{-1/2}x$ is not well defined in $\mathcal{H}^s$ because $C^{-1/2}$ is not a trace class operator. However, equation (2.30) is still well defined because the operator $C^{-1/2}$ acts only in X^N for any *fixed N*.

Notice that $C^{1/2}\zeta$ is approximately the drift term in the SPDE (1.9) and this plays a key role in obtaining the mean drift from the accept/reject mechanism; this point is elaborated on in the arguments leading up to (2.45). By (3.5) and Assumptions 3.1, 3.4 on Ψ and Ψ^N below, we will obtain a global bound on the remainder term of the form

$$(2.32) \qquad |r(x, \xi)| \le M \frac{\ell^2}{N} \|C^{1/2}\xi\|_s^2.$$

Because of our assumptions on C in (3.1), the moments of $\|C^{1/2}\xi\|_s^2$ stay uniformly bounded as $N \to \infty$. Hence, we will neglect this term to explain the heuristic ideas. Since $\xi = \sum_{i=1}^N \xi_i \phi_i$ with $\xi_i \stackrel{\mathcal{D}}{\sim} N(0, 1)$, we find that for fixed x,

$$(2.33) \qquad Q(x, \xi) \approx N\left(-\ell^2, 2\ell^2 \frac{\|\zeta\|^2}{N}\right)$$

for large N (see Lemma 5.1). Since $x \stackrel{\mathcal{D}}{\sim} \pi$, we have that $C^{-1/2}(P^N x) = \sum_{k=1}^N \rho_j \phi_j$, where ρ_j are i.i.d. $N(0, 1)$. Much as with the term $r(x, \xi)$ above, the second term in expression (2.30) for ζ can be seen as a perturbation term which is small in magnitude compared to the first term in (2.30) as $N \to \infty$. Thus, as shown in Lemma 5.2, we have $\|\zeta\|^2/N \to 1$ for π-a.e. ζ as $N \to \infty$. Returning to (2.33), this suggests that it is reasonable for N sufficiently large to make the approximation

$$(2.34) \qquad Q(x, \xi) \approx N(-\ell^2, 2\ell^2), \qquad \pi\text{-a.s.}$$

Much of this section is concerned with understanding the behavior of one step of the RWM algorithm if we make the approximation in (2.34). Once this is understood, we will retrace our steps being more careful to control the approximation error leading to (2.34).

The following lemma concerning normal random variables will be critical to identifying the source of the observed drift. It gives us the relation between the constants in the expected drift and diffusion coefficients which ensures π invariance, as will be seen later in this section.

LEMMA 2.4.   Let $Z_\ell \overset{\mathcal{D}}{\sim} N(-\ell^2, 2\ell^2)$. Then $\mathbb{P}(Z_\ell > 0) = \mathbb{E}(e^{Z_\ell} 1_{Z_\ell < 0}) = \Phi(-\ell/\sqrt{2})$ and

$$(2.35) \qquad \mathbb{E}(1 \wedge e^{Z_\ell}) = 2\Phi(-\ell/\sqrt{2}) = \beta.$$

Furthermore, if $z \overset{\mathcal{D}}{\sim} N(0, 1)$ then

$$(2.36) \qquad \mathbb{E}[z(1 \wedge e^{az+b})] = a \exp(a^2/2 + b)\Phi\left(-\frac{b}{|a|} - |a|\right)$$

for any real constants a and b.

PROOF.   A straightforward calculation. See Lemma 2 in [4]. $\square$

The calculations of the expected one step drift and diffusion needed to prove Proposition 2.1 are long and technical. In order to enhance the readability, in the next two sections we outline our proof strategy emphasizing the key calculations.

2.6. *Heuristic argument for the expected drift.*   In this section, we will give heuristic arguments which underly (2.14) from Proposition 2.1. Recall that $\{\phi_1, \phi_2, \ldots\}$ is an orthonormal basis for $\mathcal{H}$. Let $x_i^k, i \leq N$, denote the ith coordinate of x^k and C^N denote the covariance operator on X^N, the span of $\{\phi_1, \phi_2, \ldots, \phi_N\}$. Also recall that $\mathcal{F}_k$ denotes the sigma algebra generated by $\{x^n, \xi^n, \gamma^n, n \leq k\}$ and the conditional expectations $\mathbb{E}(\cdot|\mathcal{F}_k)$ are denoted by $\mathbb{E}_k(\cdot)$. Thus $\mathbb{E}_0(\cdot)$ denotes the expectation with respect to ξ^1 and γ^1 with x^0 fixed. Also, for notational convenience, set $x^0 = x$ and $\xi^1 = \xi$. Letting $\mathbb{E}_0^\xi$ denote the expectation with respect to ξ, it follows that

$$
\begin{aligned}
N\mathbb{E}_0(x_i^1 - x_i^0) &= N\mathbb{E}_0(\gamma^1(y_i^1 - x_i)) \\
&= N\mathbb{E}_0^\xi\left(\alpha(x, \xi)\sqrt{\frac{2\ell^2}{N}}(C^{1/2}\xi)_i\right) \\
&= \lambda_i\sqrt{2\ell^2 N}\mathbb{E}_0^\xi(\alpha(x, \xi)\xi_i) \\
&= \lambda_i\sqrt{2\ell^2 N}\mathbb{E}_0^\xi((1 \wedge e^{Q(x,\xi)})\xi_i).
\end{aligned}
$$

(2.37)

To approximately evaluate (2.37) using Lemma 2.4, it is easier to first factor $Q(x, \xi)$ into components involving ξ_i and those orthogonal (under $\mathbb{E}_0^{\xi}$) to them. To this end we introduce the following terms:

$$(2.38) \qquad R(x, \xi) \stackrel{\text{def}}{=} -\sqrt{\frac{2\ell^2}{N}} \sum_{j=1}^{N} \zeta_j \xi_j - \frac{\ell^2}{N} \sum_{j=1}^{N} \xi_j^2,$$

$$(2.39) \qquad R_i(x, \xi) \stackrel{\text{def}}{=} -\sqrt{\frac{2\ell^2}{N}} \sum_{j=1, j\neq i}^{N} \zeta_j \xi_j - \frac{\ell^2}{N} \sum_{j=1, j\neq i}^{N} \xi_j^2.$$

Hence, for large N (see Lemma 5.5),

$$Q(x, \xi) = R(x, \xi) - r(x, \xi) = R_i(x, \xi) - \sqrt{\frac{2\ell^2}{N}} \zeta_i \xi_i - \frac{\ell^2}{N} \xi_i^2 - r(x, \xi)$$

$$(2.40) \qquad = R_i(x, \xi) - \sqrt{\frac{2\ell^2}{N}} \zeta_i \xi_i + O\left(\frac{1}{N}\right)$$

$$\approx R_i(x, \xi) - \sqrt{\frac{2\ell^2}{N}} \zeta_i \xi_i.$$

The important observation here is that conditional on x, the random variable $R_i(x, \xi)$ is independent of ξ_i. Hence, the expectation $\mathbb{E}_0^{\xi}((1 \wedge e^{Q(x,\xi)})\xi_i)$ can be computed by first computing it over ξ_i and then over $\xi \setminus \xi_i$. Let $\mathbb{E}^{\xi_i^-}, \mathbb{E}^{\xi_i}$ denote the expectation with respect to $\xi \setminus \xi_i, \xi_i$, respectively. Using the relation (2.40), and applying (2.36) with $a = -\sqrt{\frac{2\ell^2}{N}} \zeta_i$, $z = \xi_i$ and $b = R_i(x, \xi)$, we obtain (see Lemma 5.6)

$$\mathbb{E}_0^{\xi}((1 \wedge e^{Q(x,\xi)})\xi_i)$$

$$(2.41) \qquad \approx -\sqrt{\frac{2\ell^2}{N}} \zeta_i \mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+(\ell^2/N)\zeta_i^2} \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|} - \sqrt{\frac{2\ell^2}{N}}|\zeta_i|\right)$$

$$\approx -\sqrt{\frac{2\ell^2}{N}} \zeta_i \mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2/N\zeta_i^2} \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right).$$

Now, again from the relation (2.40) and the approximation $Q(x, \xi)$ encapsulated in (2.33), it follows that for sufficiently large N

$$(2.42) \qquad R_i(x, \xi) \approx N(-\ell^2, 2\ell^2), \qquad \pi\text{-a.s.}$$

Combining (2.41) with the fact that, for large enough N, $\Phi(-R_i(x,\xi)/\sqrt{\frac{2\ell^2}{N}}|\zeta_i|) \approx$ $1_{R_i(x,\xi)<0}$, we see that Lemma 2.4 implies that (see Lemmas 5.7–5.10)

$$(2.43) \quad \mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2/N\zeta_i^2} \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N|\zeta_i|}}\right) \approx \mathbb{E}_0^{\xi_i^-}\left(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\right)$$

$$(2.44) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \approx \mathbb{E}e^{Z_\ell}1_{Z_\ell<0} = \beta/2,$$

where $Z_\ell \overset{\mathcal{D}}{\sim} N(-\ell^2, 2\ell^2)$. Hence, from (2.37), (2.41) and (2.44), we gather that for large N,

$$N\mathbb{E}_0(x_i^1 - x_i^0) \approx -\ell^2\beta\lambda_i\zeta_i.$$

To identify the drift, observe that since $C^{-1/2}$ is self-adjoint and $i \leq N$, we have $\lambda_i C^{-1/2}\phi_i = \phi_i$ and

$$(2.45) \quad \begin{aligned} \lambda_i\zeta_i &= \lambda_i\langle C^{-1/2}(P^Nx) + C^{1/2}\nabla\Psi^N(x), \phi_i\rangle \\ &= \lambda_i\langle C^{-1/2}(P^Nx) + C^{-1/2}C\nabla\Psi^N(x), \phi_i\rangle \\ &= \langle P^Nx + C^N\nabla\Psi^N(x), \phi_i\rangle. \end{aligned}$$

Hence, for large enough N, we deduce that (heuristically) the expected drift in the ith coordinate after one step of the Markov chain $\{x^k\}$ is well approximated by the expression

$$N\mathbb{E}_0(x_i^1 - x_i^0) \approx -\ell^2\beta(P^Nx + C^N\nabla\Psi^N(x))_i.$$

This is an approximation of the drift term that appears in the SPDE (1.9). Therefore, the above heuristic arguments show how the Metropolis algorithm achieves the "change of measure" by mapping π_0 to π. The above arguments can be made rigorous by quantitatively controlling the errors made. In Section 5, we quantify the size of the neglected terms and quantify the rate at which Q is well approximated by a Gaussian distribution. Using these estimates, in Section 5.2 we will retrace the arguments of this section paying attention to the cumulative error, thereby proving (2.14) of Proposition 2.1.

2.7. *Heuristic argument for the expected diffusion coefficient.* We now give the heuristic arguments for the expected diffusion coefficient, after one step of the Markov chain $\{x^k\}$. The arguments used here are much simpler than the drift calculations. The strategy is the same as in the drift case except that now we consider the covariance between two coordinates x_i^1 and x_j^1. For $1 \leq i, j \leq N$,

$$(2.46) \quad \begin{aligned} N&\mathbb{E}_0[(x_i^1 - x_i^0)(x_j^1 - x_j^0)] \\ &= N\mathbb{E}_0^\xi[(y_i^1 - x_i)(y_j^1 - x_j)\alpha(x,\xi)] \\ &= N\mathbb{E}_0^\xi[(y_i^1 - x_i)(y_j^1 - x_j)(1 \wedge \exp Q(x,\xi))] \\ &= 2\ell^2\mathbb{E}_0^\xi[(C^{1/2}\xi)_i(C^{1/2}\xi)_j(1 \wedge \exp Q(x,\xi))]. \end{aligned}$$

Now notice that

$$\mathbb{E}_0^\xi[(C^{1/2}\xi)_i(C^{1/2}\xi)_j] = \lambda_i\lambda_j\delta_{ij},$$

where $\delta_{ij} = 1_{i=j}$. Similar to the calculations used when evaluating the expected drift, we define

$$(2.47) \qquad R_{ij}(x,\xi) \overset{\text{def}}{=} -\sqrt{\frac{2\ell^2}{N}} \sum_{k=1,k\neq i,j}^N \zeta_k\xi_k - \frac{\ell^2}{N} \sum_{k=1,k\neq i,j}^N \xi_k^2$$

and observe that

$$R(x,\xi) = R_{ij}(x,\xi) - \sqrt{\frac{2\ell^2}{N}}\zeta_i\xi_i - \frac{\ell^2}{N}\xi_i^2 - \sqrt{\frac{2\ell^2}{N}}\zeta_j\xi_j - \frac{\ell^2}{N}\xi_j^2.$$

Hence, for sufficiently large N, we have $Q(x,\xi) \approx R_{ij}(x,\xi)$. By replacing $Q(x,\xi)$ in (2.46) by $R_{ij}(x,\xi)$ we can take advantage of the fact that $R_{ij}(x,\xi)$ is conditionally independent of ξ_i,ξ_j. However, the additional error term introduced is easy to estimate because the function $f(x) \overset{\text{def}}{=} (1 \wedge e^x)$ is 1-Lipschitz. So, for large enough N (Lemma 5.12),

$$\mathbb{E}_0^\xi\big[(C^{1/2}\xi)_i(C^{1/2}\xi)_j\big(1 \wedge \exp Q(x,\xi)\big)\big]$$

$$(2.48) \qquad \approx \mathbb{E}_0^\xi\big[(C^{1/2}\xi)_i(C^{1/2}\xi)_j\big(1 \wedge \exp R_{ij}(x,\xi)\big)\big]$$

$$= \lambda_i\lambda_j\delta_{ij}\mathbb{E}_0^{\xi_{ij}^-}\big[\big(1 \wedge \exp R_{ij}(x,\xi)\big)\big].$$

Again, as in the drift calculation, we have that

$$R_{ij}(x,\xi) \Longrightarrow \mathrm{N}(-\ell^2, 2\ell^2), \qquad \pi\text{-a.s.}$$

So by the dominated convergence theorem and Lemma 2.4,

$$(2.49) \qquad \lim_{N\to\infty} \mathbb{E}^{\xi_{ij}^-}\big[\big(1 \wedge \exp R_{ij}(x,\xi)\big)\big] = \beta.$$

Therefore, for large N,

$$N\mathbb{E}_0[(x_i^1 - x_i^0)(x_j^1 - x_j^0)] \approx 2\ell^2\beta\lambda_i\lambda_j\delta_{ij} = 2\ell^2\beta\langle\phi_i, C\phi_j\rangle$$

or in other words,

$$N\mathbb{E}_0[(x^1 - x^0) \otimes (x^1 - x^0)] \approx 2\ell^2\beta C^N.$$

As with the drift calculations in the last section, these calculations can be made rigorous by tracking the size of the neglected terms and quantifying the rate at which Q is approximated by the appropriate Gaussian. We will substantiate these arguments Section 5.3.

3. Main theorem. In this section we state the assumptions we make on π_0 and Ψ and then prove our main theorem.

3.1. *Assumptions on Ψ and C.*   The assumptions we make now concern (i) the rate of decay of the standard deviations in the prior or reference measure π_0 and (ii) the properties of the Radon–Nikodym derivative (likelihood function). These assumptions are naturally linked; in order for π to be well defined we require that Ψ is π_0-measurable and this can be achieved by ensuring that Ψ is continuous on a space which has full measure under π_0. In fact, in a wide range of applications, Ψ is Lipschitz on such a space [27]. In this paper we require, in addition, that Ψ be twice differentiable in order to define the diffusion limit. This, too, may be established in many applications. To avoid technicalities, we assume that $\Psi(x)$ is quadratically bounded, with first derivative linearly bounded and second derivative globally bounded. A simple example of a function Ψ satisfying the above assumptions is $\Psi(x) = \|x\|_s^2$.

ASSUMPTIONS 3.1.   The operator C and functional Ψ satisfy the following:

(1) *Decay of eigenvalues λ_i^2 of C:* There exist $M_-, M_+ \in (0, \infty)$ and $\kappa > \frac{1}{2}$ such that

$$(3.1) \qquad\qquad M_- \leq i^\kappa \lambda_i \leq M_+ \qquad \forall i \in \mathbb{Z}_+.$$

(2) *Assumptions on Ψ:* There exist constants $M_i \in \mathbb{R}, i \leq 4$ and $s \in [0, \kappa - 1/2)$ such that

$$(3.2) \qquad\qquad M_1 \leq \Psi(x) \leq M_2(1 + \|x\|_s^2) \qquad \forall x \in \mathcal{H}^s,$$

$$(3.3) \qquad\qquad \|\nabla\Psi(x)\|_{-s} \leq M_3(1 + \|x\|_s) \qquad \forall x \in \mathcal{H}^s,$$

$$(3.4) \qquad \|\partial^2\Psi(x)\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{-s})} \leq M_4 \qquad \forall x \in \mathcal{H}^s.$$

Notice also that the above assumptions on Ψ imply that for all $x, y \in \mathcal{H}^s$,

$$(3.5a) \qquad |\Psi(x) - \Psi(y)| \leq M_5(1 + \|x\|_s + \|y\|_s)\|x - y\|_s,$$

$$(3.5b) \qquad\qquad \Psi(y) = \Psi(x) + \langle\nabla\Psi(x), y - x\rangle + \text{rem}(x, y),$$

$$(3.5c) \qquad \text{rem}(x, y) \leq M_6\|x - y\|_s^2$$

for some constants $M_5, M_6 \in \mathbb{R}_+$.

REMARK 3.2.   The condition $\kappa > \frac{1}{2}$ ensures that the covariance operator for π_0 is trace class. In fact, the $\mathcal{H}^r$ norm of a realization of a Gaussian measure $N(0, C)$ defined on $\mathcal{H}$ is almost surely finite if and only if $r < \kappa - \frac{1}{2}$ [13]. Thus the choice of Sobolev space $\mathcal{H}^s$, with $s \in [0, \kappa - \frac{1}{2})$ in which we state the assumptions on Ψ, is made to ensure that the Radon–Nikodym derivative of π with respect to π_0 is well defined. Indeed, under our assumptions, Ψ is Lipschitz continuous on a set of full π_0 measure; it is hence π_0-measurable. Weaker growth assumptions on Ψ, its Lipschitz constant and second derivative could be dealt with by use of stopping time arguments.

The following lemma will be used repeatedly.

LEMMA 3.3.   *Under Assumptions* 3.1 *it follows that, for all* $a \in \mathbb{R}$,

(3.6)
$$\|C^a x\| \asymp \|x\|_{-2\kappa a}.$$

Furthermore, the function $C\nabla\Psi : \mathcal{H}^s \to \mathcal{H}^s$ *is globally Lipschitz.*

PROOF.   The first result follows from the inequality

$$\|C^a x\|^2 = \sum_{j=1}^{\infty} \lambda_j^{4a} x_j^2 \le M_+ \sum_{j=1}^{\infty} j^{-4a\kappa} x_j^2 = M_+ \|x\|_{-2\kappa a}^2,$$

and a similar lower bound, using (3.1). To prove the global Lipschitz property we first note that

(3.7)
$$\nabla\Psi(u_1) - \nabla\Psi(u_2) = K(u_1 - u_2)$$
$$:= \int_0^1 \partial^2 \Psi\big(tu_1 + (1-t)u_2\big) \, dt \, (u_1 - u_2).$$

Note that $\|K\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{-s})} \le M_4$ by (3.4). Thus,

$$\big\|C\big(\nabla\Psi(u_1) - \nabla\Psi(u_2)\big)\big\|_s$$
$$\le M \|C^{1-s/2\kappa} K(u_1 - u_2)\|$$
$$\le M \|C^{1-s/2k} K C^{s/2k} C^{-s/2k}(u_1 - u_2)\|$$
$$\le M \|C^{1-s/2\kappa} K C^{s/2k}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \|u_1 - u_2\|_s$$
$$\le M \|C^{1-s/2\kappa}\|_{\mathcal{L}(\mathcal{H}^{-s}, \mathcal{H})} \|K\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{-s})} \|C^{s/2k}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}^s)} \|u_1 - u_2\|_s.$$

The three linear operators are bounded between the appropriate spaces, in the case of $C^{1-s/2\kappa}$ by using the fact that $s < \kappa - \frac{1}{2}$ implies $s < \kappa$.   □

3.2. *Finite-dimensional approximation of the invariant distribution.*   For simplicity we assume throughout this paper that $\Psi^N(\cdot) = \Psi(P^N \cdot)$. We note again that $\nabla\Psi^N(x) = P^N \nabla\Psi(P^N x)$ and $\partial^2 \Psi^N(x) = P^N \partial^2 \Psi(P^N x) P^N$. Other approximations could be handled similarly. The function Ψ^N may be shown to satisfy the following.

ASSUMPTIONS 3.4 (Assumptions on Ψ^N).   The functions $\Psi^N : X^N \mapsto \mathbb{R}$ satisfy the same conditions imposed on Ψ given by equations (3.2), (3.3) and (3.4) with the same constants uniformly in N.

It is straightforward to show that the above assumptions on Ψ^N imply that the sequence of measures $\{\pi^N\}$ converges to π in the Hellinger metric (see [12]).

Therefore, the measures $\{\pi^N\}$ are good candidates for *finite-dimensional* approximations of π. Furthermore, the normalizing constants M_{Ψ^N} are uniformly bounded and we use this fact to obtain uniform bounds on moments of functionals in $\mathcal{H}$ under π^N.

LEMMA 3.5.   *Under the Assumptions* 3.4 *on* Ψ^N,

$$\sup_{N \in \mathbb{N}} M_{\Psi^N} < \infty$$

and for any measurable functional $f : \mathcal{H} \mapsto \mathbb{R}$, *and any* $p \geq 1$,

$$(3.8) \qquad \sup_{N \in \mathbb{N}} \mathbb{E}^{\pi^N} |f(x)|^p \leq M \mathbb{E}^{\pi_0} |f(x)|^p.$$

PROOF.   By definition,

$$M_{\Psi^N}^{-1} = \int_{\mathcal{H}} \exp\{-\Psi^N(x)\} \pi_0(dx) \geq \int_{\mathcal{H}} \exp\{-M(1 + \|x\|_s^2)\} \pi_0(dx)$$

$$\geq e^{-2M} \mathbb{P}^{\pi_0}(\|x\|_s \leq 1)$$

and therefore, if $\inf\{M_{\Psi^N}^{-1} : N \in \mathbb{N}\} > 0$, then $\sup\{M_{\Psi^N} : N \in \mathbb{N}\} < \infty$. Hence, for any $f : \mathcal{H} \mapsto \mathbb{R}$,

$$\sup_{N \in \mathbb{N}} \mathbb{E}^{\pi^N} |f(x)|^p \leq \sup_{N \in \mathbb{N}} M_{\Psi^N} \mathbb{E}^{\pi_0}\big(e^{-\Psi^N(x)} |f(x)|^p\big) \leq M \mathbb{E}^{\pi_0} |f(x)|^p$$

proving the lemma.   □

The uniform estimate given in (3.8) will be used repeatedly in the sequel.

3.3. *Statement and proof of the main theorem.*   The assumptions made above allow us to fully state the main result of this article, as outlined in Section 2.4.

THEOREM 3.6.   *Let the Assumptions* 3.1, 3.4 *hold. Let the initial condition* x^0 *of the RWM algorithm be such that* $x^0 \overset{\mathcal{D}}{\sim} \pi^N$ *and let* $z^N(t)$ *be a piecewise linear, continuous interpolant of the RWM algorithm* (2.12) *as defined in* (1.2). *Then* $z^N(t)$ *converges weakly in* $C([0, T], \mathcal{H}^s)$ *to the diffusion process* $z(t)$ *given by* (1.9) *with* $z(0) \overset{\mathcal{D}}{\sim} \pi$.

Throughout the remainder of the paper we assume that Assumptions 3.1, 3.4 hold, without explicitly stating this fact. The proof of Theorem 3.6 is given below and relies on Proposition 2.1 stated above and proved in Section 5, Proposition 2.2 stated above and proved in Section 4 and Lemma 3.7 which we now state and then prove at the end of this section.

LEMMA 3.7. *Fix any $T > 0$, any $z^0 \in \mathcal{H}^s$ and any $W \in C([0, T], \mathcal{H}^s)$. Then the integral equation* (2.23) *has a unique solution $z \in C([0, T], \mathcal{H}^s)$. Furthermore, $z = \Theta(z^0, W)$ where $\Theta: \mathcal{H}^s \times C([0, T]; \mathcal{H}^s) \to C([0, T]; \mathcal{H}^s)$ as defined in* (2.23) *is continuous.*

PROOF OF THEOREM 3.6. We begin by tracking the error in the Euler approximation argument. As before, let $x^0 \overset{\mathcal{D}}{\sim} \pi^N$ and assume $x(0) = x^0$. Returning to (2.19), using the definitions from (2.16) and Proposition 2.1, produces

$$(3.9) \qquad x^{k+1} = x^k + \mathbb{E}_k(x^{k+1} - x^k) + \sqrt{\frac{2\ell^2 \beta}{N}} \Gamma^{k+1,N},$$

$$(3.10) \qquad x^{k+1} = x^k + \frac{1}{N} d^N(x^k) + \sqrt{\frac{2\ell^2 \beta}{N}} \Gamma^{k+1,N}$$

$$(3.11) \qquad = x^k - \frac{\ell^2 \beta}{N} m^N(x^k) + \sqrt{\frac{2\ell^2 \beta}{N}} \Gamma^{k+1,N} + \frac{r^{k+1,N}}{N},$$

where $d^N(\cdot)$ is defined as in (2.25) and $r^{k+1,N}$ as in (2.17). By construction, $\mathbb{E}_k(\Gamma^{k+1,N}) = 0$ and

$$(3.12) \qquad \begin{aligned} &\mathbb{E}_k(\Gamma^{k+1,N} \otimes \Gamma^{k+1,N}) \\ &\quad = \frac{N}{2\ell^2 \beta} [\mathbb{E}_k((x^{k+1} - x^k) \otimes (x^{k+1} - x^k)) \\ &\qquad - \mathbb{E}_k(x^{k+1} - x^k) \otimes \mathbb{E}_k(x^{k+1} - x^k)] \\ &\quad = C^N + \frac{1}{2\ell^2 \beta} E^{k+1,N} - \frac{N}{2\ell^2 \beta} [\mathbb{E}_k(x^{k+1} - x^k) \otimes \mathbb{E}_k(x^{k+1} - x^k)], \end{aligned}$$

where $E^{k+1,N}$ is as given in (2.18).

Recall t^k given by (2.21) and W^N, the linear interpolant of a correctly scaled sum of the $\Gamma^{k,N}$, given by (2.22). We now define $\widehat{W}^N$ so that (2.27) holds as stated and hence, $\Theta(\widehat{W}^N) = z^N$. Define

$$r_1^N(t) \overset{\text{def}}{=} r^{k+1,N} \qquad \text{for } t \in [t^k, t^{k+1}),$$

$$r_2^N(s) \overset{\text{def}}{=} \ell^2 \beta(z^N(s) + C\nabla\Psi(z^N(s)) - m^N(\bar{z}^N(s))),$$

where $r^{k+1,N}(\cdot)$ is given by (2.17), m^N is from (2.16), $\bar{z}^N$ from (2.24) and z^N from (2.26). If

$$\widehat{W}^N(t) \overset{\text{def}}{=} W^N(t) + (1/\sqrt{2\ell^2 \beta})e^N(t)$$

with $e^N(t) = \int_0^t (r_1^N(u) + r_2^N(u))\, du$, then (2.27) holds. To see this, observe from (2.26) that

$$
\begin{aligned}
z^N(t) &= z^0 + \int_0^t d^N(\bar{z}^N(u))\, du + \sqrt{2\ell^2\beta}\, W^N(t) \\
&= z^0 - \ell^2\beta \int_0^t m^N(\bar{z}^N(u))\, du + \int_0^t r_1^N(s)\, ds + \sqrt{2\ell^2\beta}\, W^N(t) \\
&= z^0 - \ell^2\beta \int_0^t \left(z^N(u) + C\nabla\Psi(z^N(u))\right) du + \int_0^t \left(r_1^N(s) + r_2^N(s)\right) ds \\
&\quad + \sqrt{2\ell^2\beta}\, W^N(t) \\
&= z^0 - \ell^2\beta \int_0^t \left(z^N(u) + C\nabla\Psi(z^N(u))\right) du + \sqrt{2\ell^2\beta}\, \widehat{W}^N(t)
\end{aligned}
$$

and hence, with this definition of $\widehat{W}^N$, (2.27) holds.

Furthermore, we claim that

$$
\text{(3.13)} \qquad \lim_{N\to\infty} \mathbb{E}^{\pi^N}\left(\sup_{t\in[0,T]} \|e^N(t)\|_s^2 \right) = 0.
$$

To prove this, notice that

$$
\sup_{t\in[0,T]} \|e^N(t)\|_s^2 \leq M\left(\sup_{t\in[0,T]} \int_0^t \|r_1^N(u)\|_s^2\, du + \sup_{t\in[0,T]} \int_0^t \|r_2^N(u)\|_s^2\, du \right).
$$

Also

$$
\begin{aligned}
\mathbb{E}^{\pi^N} &\sup_{t\in[0,T]} \int_0^t \|r_1^N(u)\|_s^2\, du \\
&\leq \mathbb{E}^{\pi^N} \int_0^T \|r_1^N(u)\|_s^2\, du \leq M \frac{1}{N} \mathbb{E}^{\pi^N} \sum_{k=1}^N \|r^{k,N}\|_s^2 \\
&= M\mathbb{E}^{\pi^N} \|r^{1,N}\|_s^2 \xrightarrow{N\to\infty} 0,
\end{aligned}
$$

where we used stationarity of $r^{k,N}$ and (2.14) from Proposition 2.1 in the last step. We now estimate the second term similarly to complete the proof. Recall that the function $z \mapsto z + C\nabla\Psi(z)$ is Lipschitz on $\mathcal{H}^s$ by Lemma 3.3. Note also that $C^N\nabla\Psi^N(\cdot) = CP^N\nabla\Psi(P^N\cdot)$. Thus,

$$
\begin{aligned}
\|r_2^N(u)\|_s &\leq M\|z^N(u) - P^N\bar{z}^N(u)\|_s + \|C(I - P^N)\nabla\Psi(P^N\bar{z}^N(u))\|_s \\
&\leq M\left(\|z^N(u) - \bar{z}^N(u)\|_s + \|(I - P^N)\bar{z}^N(u)\|_s\right) \\
&\quad + \|(I - P^N)C\nabla\Psi(P^N\bar{z}^N(u))\|_s.
\end{aligned}
$$

But for any $u \in [t^k, t^{k+1})$, we have

$$
\|z^N(u) - \bar{z}^N(u)\|_s \leq \|x^{k+1} - x^k\|_s \leq \|y^{k+1} - x^k\|_s.
$$

This follows from the fact that $\bar{z}^N(u) = x^k$ and $z^N(u) = \frac{1}{\Delta t}((u-t^k)x^{k+1} + (t^{k+1} - u)x^k)$, because $x^{k+1} - x^k = \gamma^{k+1}(y^{k+1} - x^k)$ and $|\gamma^{k+1}| \leq 1$. For $u \in [t^k, t^{k+1})$, we also have

$$\|(P^N - I)\bar{z}^N(u)\|_s = \|(P^N - I)x^k\|_s = \|(P^N - I)x^0\|_s,$$

because x^k is not updated in $\mathcal{H}^s \setminus X^N$, and

$$\|(P^N - I)C\nabla\Psi(P^N\bar{z}^N(u))\|_s = \|(P^N - I)C\nabla\Psi(P^N x^k)\|_s.$$

Hence, we have by stationarity that, for all $u \in [0, T]$,

$$\mathbb{E}^\pi \|r_2^N(u)\|_s^2 \leq M\mathbb{E}^\pi \|y^1 - x^0\|_s^2$$
$$+ M\mathbb{E}^\pi \big(\|(P^N - I)x^0\|_s^2 + \|(P^N - I)C\nabla\Psi(P^N x^0)\|_s^2\big).$$

Equation (2.12) shows that $\mathbb{E}^\pi \|y^1 - x^0\|_s^2 \leq MN^{-1}$. The definition of P^N gives $\mathbb{E}^\pi \|(P^N - I)x\|_s^2 \leq N^{-(r-s)}\mathbb{E}^\pi \|x\|_r^2$ for any $r \in (s, \kappa - 1/2)$. Note that $\mathbb{E}^\pi \|x^0\|_r^2$ is finite for $r \in (s, \kappa - 1/2)$ by Lemma 3.5 and the properties of π_0. Similarly, we have that for $r \leq 2\kappa - s < \kappa + \frac{1}{2}$,

$$\mathbb{E}\|C\nabla\Psi(P^N x^0)\|_r^2 \leq M\mathbb{E}\|C^{1-(r+s)/2\kappa}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}\|\nabla\Psi(P^N x^0)\|_{-s}^2$$
$$\leq M\mathbb{E}(1 + \|x^0\|_s^2).$$

Hence, we deduce that $\mathbb{E}^{\pi^N} \|r_2^N(u)\|_s^2 \to 0$ uniformly for $u \in [0, T]$. It follows that

$$\mathbb{E}^{\pi^N} \sup_{t \in [0,T]} \int_0^t \|r_2^N(u)\|_s^2 \, du \leq \mathbb{E}^{\pi^N} \int_0^T \|r_2^N(u)\|_s^2 \, du \leq \int_0^T \mathbb{E}^{\pi^N} \|r_2^N(u)\|_s^2 \, du \to 0$$

and we have proved the claim concerning e^N made in (3.13).

The proof concludes with a straightforward application of the continuous mapping theorem. Let $\widehat{W}^N = W^N + \frac{1}{\sqrt{2\ell^2\beta}}e^N$. Let Ω denote the probability space generating the Markov chain in stationarity. We have shown that $e^N \to 0$ in $L^2(\Omega; C([0, T], \mathcal{H}^s))$ and by Proposition 2.2, W^N converges weakly to W a Brownian motion with covariance operator C_s in $C([0, T], \mathcal{H}^s)$. Furthermore, we also have that W is independent of z^0. Thus $(z^0, \widehat{W}^N)$ converges weakly to (z^0, W) in $\mathcal{H}^s \times C([0, T], \mathcal{H}^s)$, with z^0 and W independent. Notice that $z^N = \Theta(z^0, \widehat{W}^N)$, where Θ is defined as in Lemma 3.7. Since Θ is a continuous map by Lemma 3.7, we deduce from the continuous mapping theorem that the process z^N converges weakly in $C([0, T], \mathcal{H}^s)$ to z with law given by $\Theta(z^0, W)$. Since W is independent of z^0, this is precisely the law of the SPDE given by (1.9). □

PROOF OF LEMMA 3.7.　Consider the mapping $z^{(n)} \mapsto z^{(n+1)}$ defined by

$$z^{(n+1)}(t) = z^0 - h(\ell)\int_0^t \big(z^{(n)}(s) + C\nabla\Psi(z^{(n)}(s))\big)\,ds + \sqrt{2h(\ell)}W(t)$$

for arbitrary $z^0 \in \mathcal{H}$ and $W \in C([0, T]; \mathcal{H}^s)$. Recall from Lemma 3.3 that $z \mapsto z + C\nabla\Psi(z)$ is globally Lipschitz on $\mathcal{H}^s$. It is then a straightforward application

of the contraction mapping theorem to show that this mapping has a unique fixed point in $C([0, T]; \mathcal{H}^s)$, for T sufficiently small. Repeated application of the same idea extends this existence and uniqueness result to arbitrary time-intervals. Let z_i solve (2.23) with $(z^0, W) = (w_i, W_i)$, $i = 1, 2$. Subtracting the two equations and using the fact that $z \mapsto z + C\nabla\Psi(z)$ is globally Lipschitz on $\mathcal{H}^s$ gives

$$\|z_1(t) - z_2(t)\|_s \leq \|w_1 - w_2\|_s + M \int_0^t \|z_1(s) - z_2(s)\|_s \, ds$$

$$+ \sqrt{2\ell^2\beta}\|W_1(t) - W_2(t)\|_s.$$

Thus,

$$\sup_{0 \leq t \leq T} \|z_1(t) - z_2(t)\|_s \leq \|w_1 - w_2\|_s + M \int_0^T \sup_{0 \leq \tau \leq s} \|z_1(\tau) - z_2(\tau)\|_s \, ds$$

$$+ \sqrt{2\ell^2\beta} \sup_{0 \leq t \leq T} \|W_1(t) - W_2(t)\|_s.$$

The Gronwall lemma gives continuity in the desired spaces. $\quad\square$

4. Weak convergence of the noise process: Proof of Proposition 2.2. Throughout, we make the standing Assumptions 3.1, 3.4 without explicit mention. The proof of Proposition 2.2 uses the following result concerning triangular martingale increment arrays. The result is similar to the classical results on triangular arrays of independent increments.

Let $k_N : [0, T] \to \mathbb{Z}_+$ be a sequence of nondecreasing, right-continuous functions indexed by N with $k_N(0) = 0$ and $k_N(T) \geq 1$. Let $\{M^{k,N}, \mathcal{F}^{k,N}\}_{0 \leq k \leq k_N(T)}$ be an $\mathcal{H}^s$ valued martingale difference array. That is, for $k = 1, \ldots, k_N(T)$, we have $\mathbb{E}(M^{k,N}|\mathcal{F}^{k-1,N}) = 0$, $\mathbb{E}(\|M^{k,N}\|_s^2|\mathcal{F}^{k-1,N}) < \infty$ almost surely, and $\mathcal{F}^{k-1,N} \subset \mathcal{F}^{k,N}$. We will make use of the following result.

PROPOSITION 4.1 ([3], Proposition 5.1). *Let* $S : \mathcal{H}^s \to \mathcal{H}^s$ *be a self-adjoint, positive definite, operator with finite trace. Assume that, for all* $x \in \mathcal{H}^s, \epsilon > 0$ *and* $t \in [0, T]$, *the following limits hold in probability*:

$$(4.1) \qquad \lim_{N \to \infty} \sum_{k=1}^{k_N(T)} \mathbb{E}(\|M^{k,N}\|_s^2|\mathcal{F}^{k-1,N}) = T \operatorname{trace}(S),$$

$$(4.2) \qquad \lim_{N \to \infty} \sum_{k=1}^{k_N(t)} \mathbb{E}(\langle M^{k,N}, x\rangle_s^2|\mathcal{F}^{k-1,N}) = t\langle Sx, x\rangle_s,$$

$$(4.3) \quad \lim_{N \to \infty} \sum_{k=1}^{k_N(T)} \mathbb{E}(\langle M^{k,N}, x\rangle_s^2 1_{|\langle M^{k,N}, x\rangle_s| \geq \epsilon}|\mathcal{F}^{k-1,N}) = 0.$$

Define a continuous time process W^N *by* $W^N(t) = \sum_{k=1}^{k_N(t)} M^{k,N}$ *if* $k_N(t) \geq 1$ *and* $k_N(t) > \lim_{r \to 0_+} k_N(t - r)$, *and by linear interpolation otherwise. Then the se-*

quence of random variables W^N converges weakly in $C([0, T], \mathcal{H}^s)$ to an $\mathcal{H}^s$ valued Brownian motion W, with $W(0) = 0$, $\mathbb{E}(W(T)) = 0$, and with covariance operator S.

REMARK 4.2. The first two hypotheses of the above theorem ensure the weak convergence of finite-dimensional distributions of $W^N(t)$ using the martingale central limit theorem in $\mathbb{R}^N$; the last hypothesis is needed to verify the tightness of the family $\{W^N(\cdot)\}$. As noted in [11], the second hypothesis [equation (4.2)] of Proposition 4.1 is implied by

$$(4.4) \qquad \lim_{N \to \infty} \sum_{k=1}^{k_N(t)} \mathbb{E}(\langle M^{k,N}, e_n \rangle_s \langle M^{k,N}, e_m \rangle_s | \mathcal{F}^{k-1,N}) = t \langle S e_n, e_m \rangle_s$$

in probability, where $\{e_n\}$ is any orthonormal basis for $\mathcal{H}^s$. The third hypothesis in (4.3) is implied by the Lindeberg type condition,

$$(4.5) \qquad \lim_{N \to \infty} \sum_{k=1}^{k_N(T)} \mathbb{E}(\|M^{k,N}\|_s^2 \mathbf{1}_{\|M^{k,N}\|_s \geq \epsilon} | \mathcal{F}^{k-1,N}) = 0$$

in probability, for any fixed $\epsilon > 0$.

Using Proposition 4.1 we now give the proof of Proposition 2.2.

PROOF OF PROPOSITION 2.2. We apply Proposition 4.1 with $k_N(t) \overset{\text{def}}{=} \lfloor Nt \rfloor$, $M^{k,N} \overset{\text{def}}{=} \frac{1}{\sqrt{N}} \Gamma^{k,N}$ and $S \overset{\text{def}}{=} C_s$; the resulting definition of $W^N(t)$ from Proposition 4.1 coincides with that given in (2.22). We set $\mathcal{F}^{k,N}$ to be the sigma algebra generated by $\{x^j, \xi^j\}_{j \leq k}$ with $x^0 \sim \pi^N$. Since the chain is stationary, the noise process $\{\Gamma^{k,N}, 1 \leq k \leq N\}$ is identically distributed, and so are the errors $r^{k,N}$ and $E^{k,N}$ from (2.17) and (2.18), respectively. We now verify the three hypotheses required to apply Proposition 4.1. We generalize the notation $\mathbb{E}_0^\xi(\cdot)$ from Section 2.6 and set $\mathbb{E}^\xi(\cdot | \mathcal{F}^{k,N}) = \mathbb{E}_k^\xi(\cdot)$.

• Condition (4.1). It is enough to show that

$$\lim_{N \to \infty} \mathbb{E}^{\pi^N} \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor} \mathbb{E}_{k-1}^\xi(\|\Gamma^{k,N}\|_s^2) - \operatorname{trace}(C_s) \right| = 0$$

and condition (4.1) will follow from Markov's inequality. By (3.12) and (2.2),

$$\mathbb{E}_0^\xi(\|\Gamma^{1,N}\|_s^2) = \sum_{j=1}^N \mathbb{E}_0^\xi(\|B_s^{1/2} \Gamma^{1,N}\|^2) = \sum_{j=1}^N \mathbb{E}_0^\xi \langle \Gamma^{1,N}, B_s^{1/2} \phi_j \rangle^2$$

$$(4.6) \qquad = \sum_{j=1}^N \mathbb{E}_0^\xi \langle B_s^{1/2} \phi_j, \Gamma^{1,N} \otimes \Gamma^{1,N} B_s^{1/2} \phi_j \rangle$$

$$= \text{trace}(C_s^N) + \frac{1}{2\ell^2\beta} \sum_{j=1}^N \langle \phi_j, E^{1,N}\phi_j \rangle_s$$

(4.7)

$$- \frac{N}{2\ell^2\beta} \|\mathbb{E}_0(x^1 - x^0)\|_s^2.$$

By Proposition 2.1 it follows that $\mathbb{E}^{\pi^N}|\sum_{j=1}^N \langle \phi_j, E^{1,N}\phi_j \rangle_s| \to 0$. For the third term, notice that by Proposition 2.1 (2.14) we have

$$\mathbb{E}^{\pi^N} \frac{N}{2\ell^2\beta} \|\mathbb{E}_0(x^1 - x^0)\|_s^2 \leq M \frac{1}{N} \mathbb{E}^{\pi^N} (\|m^N(x^0)\|_s^2 + \|r^{1,N}\|_s^2)$$

(4.8)

$$\leq M \frac{1}{N} (\mathbb{E}^{\pi^N} (1 + \|x^0\|_s)^2 + \mathbb{E}^{\pi^N} \|r^{1,N}\|_s^2)$$

$$\to 0,$$

where the second inequality follows from the fact that $C\nabla\Psi$ is globally Lipschitz in $\mathcal{H}^s$. Also $\{E^{k,N}\}$ is a stationary sequence. Therefore,

$$\mathbb{E}^{\pi^N} \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor} \mathbb{E}_{k-1}^\xi (\|\Gamma^{k,N}\|_s^2) - T\,\text{trace}(C_s^N) \right|$$

$$\leq M \mathbb{E}^{\pi^N} \left(\left| \sum_{j=1}^N \langle \phi_j, E^{1,N}\phi_j \rangle_s \right| + \frac{N}{2\ell^2\beta} \|\mathbb{E}_0(x^1 - x^0)\|_s^2 \right)$$

$$+ \text{trace}(C_s^N) \left| \frac{\lfloor NT \rfloor}{N} - T \right| \to 0.$$

Condition (4.1) now follows from the fact that

$$\lim_{N\to\infty} |\text{trace}(C_s) - \text{trace}(C_s^N)| = 0.$$

• Condition (4.2). By Remark 4.2, it is enough to verify (4.4). To show (4.4), using stationarity and similar arguments used in verifying condition (4.1), it suffices to show that

(4.9) $$\lim_{N\to\infty} \mathbb{E}^{\pi^N} |\mathbb{E}_0^\xi (\langle \Gamma^{1,N}, \widehat{\phi}_n \rangle_s \langle \Gamma^{1,N}, \widehat{\phi}_m \rangle_s) - \langle \widehat{\phi}_n, C_s^N \widehat{\phi}_m \rangle_s| = 0,$$

where $\{\widehat{\phi}_k\}$ is as defined in (2.7). We have

$$\mathbb{E}^{\pi^N} |\mathbb{E}_0^\xi (\langle \Gamma^{1,N}, \widehat{\phi}_n \rangle_s \langle \Gamma^{1,N}, \widehat{\phi}_m \rangle_s) - \langle \widehat{\phi}_n, C_s^N \widehat{\phi}_m \rangle_s|$$

$$= n^{-s} m^{-s} \mathbb{E}^{\pi^N} |\mathbb{E}_0^\xi (\langle \Gamma^{1,N}, \phi_n \rangle_s \langle \Gamma^{1,N}, \phi_m \rangle_s) - \langle \phi_n, C_s^N \phi_m \rangle_s|$$

and therefore, it is enough to show that

(4.10) $$\lim_{N\to\infty} \mathbb{E}^{\pi^N} |\mathbb{E}_0^\xi (\langle \Gamma^{1,N}, \phi_n \rangle_s \langle \Gamma^{1,N}, \phi_m \rangle_s) - \langle \phi_n, C_s^N \phi_m \rangle_s| = 0.$$

Indeed we have

$$
\begin{aligned}
\langle \Gamma^{1,N}, \phi_n \rangle_s \langle \Gamma^{1,N}, \phi_m \rangle_s &= \langle \Gamma^{1,N}, B_s \phi_n \rangle \langle \Gamma^{1,N}, B_s \phi_m \rangle \\
&= \langle B_s \phi_n, \Gamma^{1,N} \otimes \Gamma^{1,N} B_s \phi_m \rangle \\
&= \langle \phi_n, B_s^{1/2} \Gamma^{1,N} \otimes \Gamma^{1,N} B_s^{1/2} \phi_m \rangle_s
\end{aligned}
$$

and from (3.12) and Proposition 2.1 we obtain

$$
\begin{aligned}
&\langle \phi_n, B_s^{1/2} \Gamma^{1,N} \otimes \Gamma^{1,N} B_s^{1/2} \phi_m \rangle_s - \langle \phi_n, C_s^N \phi_m \rangle_s \\
&= \langle \phi_n, B_s^{1/2} \Gamma^{1,N} \otimes \Gamma^{1,N} B_s^{1/2} \phi_m \rangle_s - \langle \phi_n, B_s^{1/2} C^N B_s^{1/2} \phi_m \rangle_s \\
&= n^s m^s \langle \phi_n, E^{1,N} \phi_m \rangle_s - \frac{N}{2\ell^2 \beta} \mathbb{E}_0(\langle x^1 - x^0, \phi_n \rangle_s) \mathbb{E}_0(\langle x^1 - x^0, \phi_m \rangle_s).
\end{aligned}
$$

From Proposition 2.1, it follows that $\lim_{N \to \infty} \mathbb{E}^{\pi^N} |\langle \phi_n, E^{1,N} \phi_m \rangle_s| = 0$. Also notice that

$$
\begin{aligned}
&N^2 [\mathbb{E}^{\pi^N} |\mathbb{E}_0(\langle x^1 - x^0, \phi_n \rangle_s) \mathbb{E}_0(\langle x^1 - x^0, \phi_m \rangle_s)|]^2 \\
&\leq M \mathbb{E}^{\pi^N} \big(N \|\mathbb{E}_0(x^1 - x^0)\|_s^2 \|\phi_n\|_s^2 \big) \mathbb{E}^{\pi^N} \big(N \|\mathbb{E}_0(x^1 - x^0)\|_s^2 \|\phi_m\|_s^2 \big) \\
&\to 0
\end{aligned}
$$

by the calculation done in (4.8). Thus (4.10) holds and since $|\langle \phi_n, C_s \phi_m \rangle_s - \langle \phi_n, C_s^N \phi_m \rangle_s| \to 0$, equation (4.2) follows from Markov's inequality.

- Condition (4.3). From Remark 4.2 it follows that verifying (4.5) suffices to establish (4.3).

  To verify (4.5), notice that for any $\epsilon > 0$,

$$
\begin{aligned}
\mathbb{E}^{\pi^N} \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor} \mathbb{E}_{k-1}^{\xi} \big(\|\Gamma^{k,N}\|_s^2 \mathbf{1}_{\{\|\Gamma^{k,N}\|_s^2 \geq \epsilon N\}} \big) \right| \\
\leq \frac{\lfloor NT \rfloor}{N} \mathbb{E}^{\pi^N} \big(\|\Gamma^{1,N}\|_s^2 \mathbf{1}_{\{\|\Gamma^{1,N}\|_s^2 \geq \epsilon N\}} \big) \to 0
\end{aligned}
$$

by the dominated convergence theorem since

$$
\lim_{N \to \infty} \mathbb{E}^{\pi^N} \|\Gamma^{1,N}\|_s^2 = \operatorname{trace}(C_s) < \infty.
$$

Thus (4.5) is verified.

Thus we have verified all three hypotheses of Proposition 4.1, proving that $W^N(t)$ converges weakly to $W(t)$ in $C([0, T]; \mathcal{H}^s)$.

Recall that $X^R \subset \mathcal{H}^s$ denotes the R-dimensional subspace $P^R \mathcal{H}^s$. To prove the second claim of Proposition 2.2, we need to show that $(x^0, W^N(t))$ converges weakly to $(z^0, W(t))$ in $(\mathcal{H}^s, C([0, T]; \mathcal{H}^s))$ as $N \to \infty$ where $z^0 \sim \pi$ and z^0 is independent of the limiting noise W. For showing this, it is enough to show that

for any $R \in \mathbb{N}$, the pair $(x^0, P^R W^N(t))$ converges weakly to (z^0, Z_R) for every $t > 0$, where Z_R is a Gaussian random variable on X^R with mean zero, covariance $t P^R C_s P^R$ and independent of z^0. We will prove this statement as the corollary of the following lemma.

LEMMA 4.3.   *Let* $x^0 \sim \pi^N$ *and let* $\{\theta^{k,N}\}$ *be any stationary martingale sequence adapted to the filtration* $\{\mathcal{F}^{k,N}\}$ *and furthermore, assume that there exists a stationary sequence* $\{U^{k,N}\}$ *such that for all* $k \geq 1$ *and any* $u \in X^R$:

(1) $\mathbb{E}_{k-1}^{\xi} |\langle u, P^R \theta^{k,N} \rangle_s|^2 = \langle u, P^R C_s u \rangle_s + U^{k,N}$, $\lim_{N \to \infty} \mathbb{E}^{\pi^N} |U^{1,N}| = 0$.
(2) $\mathbb{E}_{k-1}^{\xi} \|\theta^{k,N}\|_s^3 \leq M$.

Then for any $\mathbf{t} \in \mathcal{H}^s$, $u \in X^R$, $R \in \mathbb{N}$ *and* $t > 0$,

(4.11)
$$\lim_{N \to \infty} \mathbb{E}^{\pi^N} \left(e^{i \langle \mathbf{t}, x^0 \rangle_s + (i/\sqrt{N}) \sum_{k=1}^{\lfloor Nt \rfloor} \langle u, P^R \theta^{k,N} \rangle_s} \right)$$
$$= \mathbb{E}^{\pi} \left(e^{i \langle \mathbf{t}, z^0 \rangle_s - (t/2) \langle u, P^R C_s u \rangle_s} \right).$$

Note: Here and in Corollary 4.4, $i = \sqrt{-1}$.

PROOF OF LEMMA 4.3.   We show (4.11) for $t = 1$, since the calculations are nearly identical for an arbitrary t with minor notational changes. Indeed, we have

$$\mathbb{E}^{\pi^N} \left(e^{i \langle \mathbf{t}, x^0 \rangle_s + (i/\sqrt{N}) \sum_{k=1}^{N} \langle u, P^R \theta^{k,N} \rangle_s} \right)$$
$$= \mathbb{E}^{\pi^N} \left(\mathbb{E}_{N-1}^{\xi} \left(e^{i \langle \mathbf{t}, x^0 \rangle_s + (i/\sqrt{N}) \sum_{k=1}^{N} \langle u, P^R \theta^{k,N} \rangle_s} \right) \right).$$

By Taylor's expansion,

(4.12)
$$\mathbb{E}^{\pi^N} \left(\mathbb{E}_{N-1}^{\xi} \left(e^{i \langle \mathbf{t}, x^0 \rangle_s + (i \sqrt{N}) \sum_{k=1}^{N} \langle u, P^R \theta^{k,N} \rangle_s} \right) \right)$$
$$= \mathbb{E} \left[e^{i \langle \mathbf{t}, x^0 \rangle_s + (i/\sqrt{N}) \sum_{k=1}^{N-1} \langle u, P^R \theta^{k,N} \rangle_s} \right.$$
$$\times \left(1 - \frac{1}{2N} \mathbb{E}_{N-1}^{\xi} |\langle u, P^R \theta^{N,N} \rangle_s|^2 \right.$$
$$\left. \left. + M \left(\frac{1}{N^{3/2}} V^N \wedge 2 \right) \right) \right],$$

where $|V^N| \leq \mathbb{E}_{N-1}^{\xi} |\langle u, P^R \theta^{N,N} \rangle_s|^3 \leq M$, since by assumption $\mathbb{E}_{N-1}^{\xi} \|\theta^{N,N}\|_s^3 \leq M$. We also have that

$$\mathbb{E}_{N-1}^{\xi} |\langle u, P^R \theta^{N,N} \rangle_s|^2 = \langle u, P^R C_s u \rangle_s + U^{N,N},$$
$$\lim_{N \to \infty} \mathbb{E}^{\pi^N} |U^{N,N}| = 0.$$

Thus from (4.12) we deduce that

$$\mathbb{E}^{\pi^N}\big(e^{i\langle \mathbf{t},x^0\rangle_s+(i/\sqrt{N})\sum_{k=1}^N\langle u,P^R\theta^{k,N}\rangle_s}\big)$$

$$= \mathbb{E}^{\pi^N}\bigg[e^{i\langle \mathbf{t},x^0\rangle_s+(i/\sqrt{N})\sum_{k=1}^{N-1}\langle u,P^R\theta^{k,N}\rangle_s}\bigg(1-\frac{1}{2N}\langle u,P^RC_su\rangle_s\bigg)\bigg]$$

(4.13) $+\, S^N,$

$$|S^N| \le M\mathbb{E}^{\pi^N}\bigg(\frac{1}{2N}|U^{N,N}|+\frac{1}{N^{3/2}}|V^N|\bigg)$$

$$= M\frac{1}{N}\mathbb{E}^{\pi^N}\bigg(|U^{N,N}|+\frac{1}{\sqrt{N}}\bigg).$$

Proceeding recursively we obtain

$$\mathbb{E}^{\pi^N}\big(e^{i\langle \mathbf{t},x^0\rangle_s+(i/\sqrt{N})\sum_{k=1}^N\langle u,P^R\theta^{k,N}\rangle_s}\big)$$

$$= \mathbb{E}^{\pi^N}\bigg[e^{i\langle \mathbf{t},x^0\rangle_s}\bigg(1-\frac{1}{2N}\langle u,P^RC_su\rangle_s\bigg)^N\bigg]+\sum_{k=1}^N S^k.$$

By the stationarity of $\{U^{k,N}\}$ and the fact that $\mathbb{E}^\pi|U^{k,N}| \to 0$ as $N\to\infty$, from (4.13) it follows that

$$\sum_{k=1}^N|S^k| \le M\sum_{k=1}^N\frac{1}{N}\bigg(\mathbb{E}^{\pi^N}|U^k|+\frac{1}{\sqrt{N}}\bigg) \le M\bigg(\mathbb{E}^{\pi^N}|U^1|+\frac{1}{\sqrt{N}}\bigg)\to 0.$$

Thus we have shown that

$$\mathbb{E}^{\pi^N}\bigg[e^{i\langle \mathbf{t},x^0\rangle_s}\bigg(1-\frac{1}{2N}\langle u,P^RC_su\rangle_s\bigg)^N\bigg] = \mathbb{E}^{\pi^N}\big[e^{i\langle \mathbf{t},x^0\rangle_s-(1/2)\langle u,P^RC_su\rangle_s}\big]+o(1),$$

and the result follows from the fact that $\mathbb{E}^{\pi^N}[e^{i\langle \mathbf{t},x^0\rangle_s}] \to \mathbb{E}^\pi[e^{i\langle \mathbf{t},z^0\rangle_s}]$, finishing the proof of Lemma 4.3. $\square$

As a corollary of Lemma 4.3, we obtain the following.

COROLLARY 4.4. *The pair* (x^0, W^N) *converges weakly to* (z^0, W) *in* $C([0,$ $T];\mathcal{H}^s)$ *where* W *is a Brownian motion with covariance operator* C_s *and is independent of* z^0 *almost surely.*

PROOF.   As mentioned before, it is enough to show that for any $\mathbf{t}\in\mathcal{H}^s$, $u\in X^R$, $R\in\mathbb{N}$ and $t>0$,

(4.14)
$$\lim_{N\to\infty}\mathbb{E}^{\pi^N}\big(e^{i\langle \mathbf{t},x^0\rangle_s+(i/\sqrt{N})\sum_{k=1}^{\lfloor Nt\rfloor}\langle u,P^R\Gamma^{k,N}\rangle_s}\big)$$

$$= \mathbb{E}^\pi\big(e^{i\langle \mathbf{t},z^0\rangle_s-(t/2)\langle u,P^RC_su\rangle_s}\big).$$

Now we verify the conditions of Lemma 4.3 to show (4.14). To verify the first hypothesis of Lemma 4.3, notice that from Proposition 2.1 we obtain that for $k \geq 1$,

$$\mathbb{E}_{k-1}^{\xi}|\langle u, P^R \Gamma^{k,N} \rangle_s|^2 = \mathbb{E}_{k-1}^{\xi}\langle B_s u, P^R \Gamma^{k,N} \otimes \Gamma^{k,N} B_s u \rangle$$

$$= \langle u, P^R C_s u \rangle_s + U^{k,N},$$

$$|U^{k,N}| \leq \frac{1}{2\ell^2\beta} M \sum_{l,j=1}^{R \wedge N} u_l u_j |\langle \phi_l, P^M E^{k,N} \phi_j \rangle_s|$$

$$+ \frac{N}{2\ell^2\beta} \|\mathbb{E}_{k-1}^{\xi}(x^k - x^{k-1})\|_s^2 \|u\|_s^2$$

$$+ |\langle u, P^R C_s^N u \rangle_s - \langle u, P^R C_s u \rangle_s|,$$

where $\{E^{k,N}\}$ is as defined in (2.18). Because $\{\Gamma^{k,N}\}$ is stationary, we deduce that $\{U^{k,N}\}$ is stationary. From Proposition 2.1 we obtain

$$\lim_{N \to \infty} \sum_{l,j=1}^{R \wedge N} \mathbb{E}^{\pi^N} |\langle \phi_l, P^M E^{k,N} \phi_j \rangle_s| = 0$$

and $\mathbb{E}^{\pi^N} \frac{N}{2\ell^2\beta} \|\mathbb{E}_{k-1}^{\xi}(x^k - x^{k-1})\|_s^2 \to 0$ by the calculation in (4.8). Thus we have shown that $\mathbb{E}^{\pi}|U^{1,N}| \to 0$ as $N \to \infty$. The second hypothesis of Lemma 4.3 is easily verified since $\mathbb{E}_{k-1}^{\xi}\|\Gamma^{k,N}\|_s^3 \leq M \mathbb{E}_{k-1}^{\xi}\|C^{1/2}\xi^k\|_s^3 \leq M$. Thus the corollary follows from Lemma 4.3. □

Thus we have shown that (x^0, W^N) converges weakly to (z^0, W) where W is a Brownian motion in $\mathcal{H}^s$ with covariance operator C_s, and by the above corollary we see that W is independent of x^0 almost surely, proving the two claims made in Proposition 2.2 and the proof is complete. □

5. Mean drift and diffusion: Proof of Proposition 2.1.

To prove this key proposition we make the standing Assumptions 3.1, 3.4 from Section 3.1 without explicit statement of this fact within the individual lemmas. We start with several preliminary bounds and then consider the drift and diffusion terms, respectively.

5.1. *Preliminary estimates.*   Recall the definitions of $R(x, \xi)$, $R_i(x, \xi)$ and $R_{ij}(x, \xi)$ from equations (2.38), (2.39) and (2.47), respectively. These quantities were introduced so that the term in the exponential of the acceptance probability $Q(x, \xi)$ could be replaced with $R_i(x, \xi)$ and $R_{ij}(x, \xi)$ to take advantage of the fact that, conditional on x, $R_i(x, \xi)$ is independent of ξ_i and $R_{ij}(x, \xi)$ is independent of ξ_i, ξ_j. In the next lemma, we estimate the additional error due to this replacement of $Q(x, \xi)$. Recall that $\mathbb{E}_0^{\xi}$ denotes expectation with respect to $\xi = \xi_0$ as in Section 2.2.

LEMMA 5.1.

$$(5.1) \qquad \mathbb{E}_0^\xi |Q(x,\xi) - R_i(x,\xi)|^2 \le \frac{M}{N}(1 + |\zeta_i|^2),$$

$$(5.2) \qquad \mathbb{E}_0^\xi (Q(x,\xi) - R_{ij}(x,\xi))^2 \le \frac{M}{N}(1 + |\zeta_i|^2 + |\zeta_j|^2).$$

PROOF. Since ξ_j are i.i.d. N(0, 1), using (2.1) and (3.1), we obtain that

$$(5.3) \qquad \mathbb{E}\|C^{1/2}\xi\|_s^4 \le 3(\mathbb{E}\|C^{1/2}\xi\|_s^2)^2 \le M\left(\sum_{j=1}^{\infty} j^{2s-2k}\right)^2 < \infty$$

since $s < k - \frac{1}{2}$.

Starting from (2.40), the estimates in (2.32) and (5.3) imply that

$$\mathbb{E}_0^\xi |Q(x,\xi) - R_i(x,\xi)|^2 \le M\left(\mathbb{E}_0^\xi |r(x,\xi)|^2 + \frac{1}{N}\mathbb{E}_0^\xi \zeta_i^2 \xi_i^2 + \frac{1}{N^2}\mathbb{E}\xi_i^4\right)$$

$$\le M\left(\frac{1}{N^2}\mathbb{E}\|C^{1/2}\xi\|_s^4 + \frac{1}{N}\zeta_i^2 + \frac{3}{N^2}\right)$$

$$\le M\frac{1}{N}(1 + \zeta_i^2)$$

verifying the first part of the lemma. A very similar argument for the second part finishes the proof. $\square$

The random variables $R(x,\xi)$, $R_i(x,\xi)$ and $R_{ij}(x,\xi)$ are approximately Gaussian random variables. Indeed it can be readily seen that

$$R(x,\xi) \approx N\left(-\ell^2, 2\frac{\ell^2}{N}\|\zeta\|^2\right).$$

The next lemma contains a crucial observation. We show that the sequence of random variables $\{\frac{\|\zeta\|^2}{N}\}$ converges to 1 almost surely under both π_0 and π. Thus $R(x,\xi)$ converges almost surely to $Z_\ell \overset{\text{def}}{=} N(-\ell^2, 2\ell^2)$ and thus the expected acceptance probability $\mathbb{E}\alpha(x,\xi) = 1 \wedge e^{Q(x,\xi)}$ converges to $\beta = \mathbb{E}(1 \wedge e^{Z_\ell})$.

LEMMA 5.2. As $N \to \infty$ we have

$$(5.4) \qquad \frac{1}{N}\|\zeta\|^2 \to 1, \qquad \pi_0\text{-a.s.} \quad and \quad \frac{1}{N}\|\zeta\|^2 \to 1, \qquad \pi\text{-a.s.}$$

Furthermore, for any $m \in \mathbb{N}$, $\alpha \ge 2$, $s < \kappa - \frac{1}{2}$ and for any $c \ge 0$,

$$(5.5) \qquad \limsup_{N \in \mathbb{N}} \mathbb{E}^{\pi^N} \sum_{j=1}^{N} \lambda_j^\alpha j^{2s} |\zeta_j|^m e^{(c/N)\|\zeta\|^2} < \infty.$$

Finally, we have

$$(5.6) \qquad \lim_{N \to \infty} \mathbb{E}^{\pi^N}\left(\left|1 - \frac{1}{N}\|\zeta\|^2\right|^2\right) = 0.$$

PROOF.   The proof proceeds by showing the conclusions first in the case when $x \overset{\mathcal{D}}{\sim} \pi_0$; this is easier because the finite-dimensional distributions are Gaussian and by Fernique's theorem x has exponential moments. Next we notice that the almost sure properties are preserved under the change of measure π. To show the convergence of moments, we use our hypothesis that the Radon–Nikodym derivative $\frac{d\pi^N}{d\pi_0}$ is bounded from above independently of N, as shown in Lemma 3.5, equation (3.8).

Indeed, first let $x \overset{\mathcal{D}}{\sim} \pi_0$. Recall that $\zeta = C^{-1/2}(P^N x) + C^{1/2}\nabla\Psi^N(x)$ and

$$(5.7) \qquad \|\nabla\Psi^N(x)\|_{-s} \le M_3(1 + \|x\|_s).$$

Using (3.6) and the fact that $s < \kappa - \frac{1}{2}$ so that $-\kappa < -s$, we deduce that

$$
\begin{aligned}
\|C^{1/2}\nabla\Psi^N(x)\| &\asymp \|\nabla\Psi^N(x)\|_{-\kappa} \\
&\le \|\nabla\Psi^N(x)\|_{-s} \\
&\le M(1 + \|x\|_s)
\end{aligned}
$$

uniformly in N. Also, since x is Gaussian under π_0, from (2.4), we may write $C^{-1/2}(P^N x) = \sum_{k=1}^{N} \rho_k \phi_k$, where ρ_k are i.i.d. N(0, 1). Note that

$$(5.8) \qquad
\begin{aligned}
\frac{1}{N}\|\zeta\|^2 &= \frac{1}{N}\|C^{-1/2}(P^N x) + C^{1/2}\nabla\Psi^N(x)\|^2 \\
&= \frac{1}{N}\big(\|C^{-1/2}(P^N x)\|^2 + 2\langle C^{-1/2}(P^N x), C^{1/2}\nabla\Psi^N(x)\rangle \\
&\qquad\qquad\qquad\qquad\qquad + \|C^{1/2}\nabla\Psi^N(x)\|^2\big) \\
&= \frac{1}{N}\big(\|C^{-1/2}(P^N x)\|^2 + 2\langle P^N x, \nabla\Psi^N(x)\rangle + \|C^{1/2}\nabla\Psi^N(x)\|^2\big) \\
&= \frac{1}{N}\sum_{k=1}^{N} \rho_k^2 + \gamma,
\end{aligned}
$$

where

$$(5.9) \qquad
\begin{aligned}
|\gamma| &\le \frac{1}{N}\big(2\|x\|_s\|\nabla\Psi^N(x)\|_{-s} + \|C^{1/2}\nabla\Psi^N(x)\|^2\big) \\
&\le \frac{M}{N}\big(2\|x\|_s(1 + \|x\|_s) + (1 + \|x\|_s)^2\big).
\end{aligned}
$$

Under π_0, we have $\|x\|_s < \infty$ a.s., for $s < \kappa - \frac{1}{2}$ and hence, by (5.9), we conclude that $|\gamma| \to 0$ almost surely as $N \to \infty$. Now, by the strong law of large numbers, $\frac{1}{N} \sum_{k=1}^{N} \rho_k^2 \to 1$ almost surely. Hence, from (5.8) we obtain that under π_0, $\lim_{N \to \infty} \frac{1}{N} \|\zeta\|^2 = 1$ almost surely, proving the first equation in (5.4). Now the second equation in (5.4) follows by noting that almost sure limits are preserved under a (absolutely continuous) change of measure.

Next, notice that by (5.8) and the Cauchy–Schwarz inequality, for any $c > 0$,

$$\big(\mathbb{E}^{\pi_0} e^{(c/N)\|\zeta\|^2}\big)^2 \le \big(\mathbb{E}^{\pi_0} e^{(2c/N) \sum \rho_k^2}\big)\big(\mathbb{E}^{\pi_0} e^{2c\gamma}\big)$$

$$\le \big(\mathbb{E}^{\pi_0} e^{(2c/N) \sum \rho_k^2}\big)\big(\mathbb{E}^{\pi_0} e^{(M/N)\|x\|_s^2}\big).$$

Using the fact that $\sum_{k=1}^{N} \rho_k^2$ has chi-squared distribution with N degrees of freedom gives

$$(5.10) \quad \big(\mathbb{E}^{\pi_0} e^{(c/N)\|\zeta\|^2}\big)^2 \le M e^{-(N/2)\log(1-4c/N)} \big(\mathbb{E}^{\pi_0} e^{(M/N)\|x\|_s^2}\big) \le M,$$

where the last inequality follows from Fernique's theorem since $\mathbb{E}^{\pi_0} e^{(M/N)\|x\|_s^2} < \infty$ for sufficiently large N. Hence, by applying Lemma 3.5, equation (3.8), it follows that $\limsup_{N \to \infty} \mathbb{E}^{\pi^N} e^{(c/N)\|\zeta\|^2} < \infty$. Notice that we also have the bound

$$|\zeta_k|^m \le M\big(|\rho_k|^m + |\lambda_k|^m(1 + \|x\|_s^m)\big).$$

Since $s < k - 1/2$, we have that $\sum_{j=1}^{\infty} \lambda_j^2 j^{2s} < \infty$ and therefore, it follows that for $\alpha \ge 2$,

$$(5.11) \qquad \limsup_{N \to \infty} \sum_{k=1}^{N} (\mathbb{E}^{\pi^N} \lambda_k^{2\alpha} j^{2s} |\zeta_k|^{2m})^{1/2} < \infty.$$

Hence the claim in (5.5) follows from applying Cauchy–Schwarz combined with (5.10) and (5.11). Similarly, a straightforward calculation yields that $\mathbb{E}^{\pi^0}(|1 - \frac{1}{N}\|\zeta\|^2|^2) \le \frac{M}{N}$. Hence, again by Lemma 3.5,

$$\lim_{N \to \infty} \mathbb{E}^{\pi^N}\left(\left|1 - \frac{1}{N}\|\zeta\|^2\right|^2\right) = 0$$

proving the last claim and the proof is complete. $\quad \square$

Recall that $Q(x, \xi) = R(x, \xi) - r(x, \xi)$. Thus, from (2.32) and Lemma 5.1 it follows that $R_i(x, \xi)$ and $R_{ij}(x, \xi)$ also are approximately Gaussian. Therefore, the conclusion of Lemma 5.2 leads to the reasoning that, for any fixed realization of $x \overset{\mathcal{D}}{\sim} \pi$, the random variables $R(x, \xi)$, $R_i(x, \xi)$ and $R_{ij}(x, \xi)$ all converge to the same weak limit $Z_\ell \sim \mathrm{N}(-\ell^2, 2\ell^2)$ as the dimension of the noise ξ goes to ∞. In the rest of this subsection, we rigorize this argument by deriving a Berry–Essen bound for the weak convergence of $R(x, \xi)$ to Z_ℓ.

For this purpose, it is natural and convenient to obtain these bounds in the Wasserstein metric. Recall that the Wasserstein distance between two random variables $\text{Wass}(X, Y)$ is defined by

$$\text{Wass}(X, Y) \stackrel{\text{def}}{=} \sup_{f \in \text{Lip}_1} \mathbb{E}(f(X) - f(Y)),$$

where Lip_1 is the class of 1-Lipschitz functions. The following lemma gives a bound for the Wasserstein distance between $R(x, \xi)$ and Z_ℓ.

LEMMA 5.3.   *Almost surely with respect to* $x \sim \pi$,

$$(5.12) \qquad \text{Wass}(R(x, \xi), Z_\ell) \leq M \left(\frac{1}{N^{3/2}} \sum_{j=1}^{N} |\zeta_j|^3 + \left| 1 - \frac{\|\zeta\|^2}{N} \right| + \frac{1}{\sqrt{N}} \right),$$

$$(5.13) \quad \text{Wass}(R(x, \xi), R_i(x, \xi)) \leq \frac{M}{\sqrt{N}} (|\zeta_i| + 1).$$

PROOF.   Define the Gaussian random variable $G \stackrel{\text{def}}{=} -\sqrt{\frac{2\ell^2}{N}} \sum_{k=1}^{N} \zeta_k \xi_k - \ell^2$. For any 1-Lipschitz function f,

$$\left| \mathbb{E}^\xi (f(G) - f(R(x, \xi))) \right| \leq \ell^2 \mathbb{E}^\xi \left| 1 - \frac{1}{N} \sum_{k=1}^{N} \xi_k^2 \right| < M \frac{1}{\sqrt{N}}$$

implying that $\text{Wass}(G, R(x, \xi)) \leq M \frac{1}{\sqrt{N}}$. Now, from classical Berry–Esseen estimates (see [26]), we have that

$$\text{Wass}(G, Z_\ell) \leq M \frac{1}{N^{3/2}} \sum_{j=1}^{N} |\zeta_j|^3 + M \left| 1 - \frac{\|\zeta\|^2}{N} \right|.$$

Hence the proof of the first claim follows from the triangle inequality. To see the second claim, notice that for any 1-Lipschitz function f we have

$$\mathbb{E}_0^\xi |f(R(x, \xi)) - f(R_i(x, \xi))| \leq \mathbb{E}_0^\xi |R(x, \xi) - R_i(x, \xi)| \leq M \frac{1}{\sqrt{N}} (1 + |\zeta_i|)$$

and the proof is complete. $\square$

Hence, from equations (5.13) and (5.12), we obtain

$$(5.14) \quad \begin{aligned} &\text{Wass}(R_i(x, \xi), Z_\ell) \\ &\leq M \left(\frac{1}{\sqrt{N}} (|\zeta_i| + 1) + \frac{1}{N^{3/2}} \sum_{j=1}^{N} |\zeta_j|^3 + \left| 1 - \frac{\|\zeta\|^2}{N} \right| \right). \end{aligned}$$

We conclude this section with the following observation which will be used later. Recall the Kolmogorov–Smirnov (KS) distance between two random variables (W, Z):

$$(5.15) \qquad \mathrm{KS}(W, Z) \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}} |\mathbb{P}(W \le t) - \mathbb{P}(Z \le t)|.$$

LEMMA 5.4. *If a random variable Z has a density with respect to the Lebesgue measure, bounded by a constant M, then*

$$(5.16) \qquad \mathrm{KS}(W, Z) \le \sqrt{4M \operatorname{Wass}(W, Z)}.$$

We could not find the reference for the above in any published literature, so we include a short proof here which was taken from the unpublished lecture notes [10].

PROOF OF LEMMA 5.4.   Fix $t \in \mathbb{R}$ and $\epsilon > 0$. Define two functions g_1 and g_2 as $g_1(y) = 1$ for $y \in (-\infty, t)$, $g_1(y) = 0$ for $y \in [t + \epsilon, \infty)$ and linear interpolation in between. Similarly, define $g_2(y) = 1$, for $y \in (-\infty, t - \epsilon]$, $g_2(y) = 0$, for $y \in [t, \infty)$ and linear interpolation in between. Then g_1 and g_2 form upper and lower envelopes for the function $1_{(-\infty, t]}(y)$. So

$$\mathbb{P}(W \le t) - \mathbb{P}(Z \le t) \le \mathbb{E}g_1(W) - \mathbb{E}g_1(Z) + \mathbb{E}g_1(Z) - \mathbb{P}(Z \le T).$$

Since g_1 is $\frac{1}{\epsilon}$-Lipschitz, we have $\mathbb{E}g_1(W) - \mathbb{E}g_1(Z) \le \frac{1}{\epsilon} \operatorname{Wass}(W, Z)$ and $\mathbb{E}g_1(Z) - \mathbb{P}(Z \le t) \le M\epsilon$ since Z has density bounded by M. Similarly, using the function g_2, it follows that the same bound holds for the difference $\mathbb{P}(Z \le t) - \mathbb{P}(W \le t)$. Optimizing over ϵ yields the required bound.   □

5.2. *Rigorous estimates for the drift*: *Proof of Proposition* 2.1, *equation* (2.14). In the following series of lemmas we retrace the arguments from Section 2.6 while deriving explicit bounds for the error terms. Lemma 5.11 at the end of the section gives control of the error terms.

The following lemma shows that $Q(x, \xi)$ is well approximated by $R_i(x, \xi) - \sqrt{\frac{2\ell^2}{N}}\zeta_i \xi_i$, as indicated in (2.40).

LEMMA 5.5.

$$N\mathbb{E}_0(x_i^1 - x_i) = \lambda_i \sqrt{2\ell^2 N} \mathbb{E}_0^\xi \big((1 \wedge e^{R_i(x, \xi) - \sqrt{2\ell^2/N}\zeta_i \xi_i})\xi_i\big) + \omega_0(i),$$

$$|\omega_0(i)| \le \frac{M}{\sqrt{N}}\lambda_i.$$

PROOF.   We have

$$N\mathbb{E}_0(x_i^1 - x_i^0) = N\mathbb{E}_0(\gamma^0(y_i^0 - x_i)) = N\mathbb{E}_0^\xi\left(\alpha(x, \xi)\sqrt{\frac{2\ell^2}{N}}(C^{1/2}\xi)_i\right)$$

$$= \lambda_i \sqrt{2\ell^2 N} \mathbb{E}_0^\xi(\alpha(x, \xi)\xi_i) = \lambda_i \sqrt{2\ell^2 N}\mathbb{E}_0^\xi\big((1 \wedge e^{Q(x, \xi)})\xi_i\big).$$

Now we observe that

$$\mathbb{E}_0^\xi\big((1 \wedge e^{Q(x,\xi)})\xi_i\big) = \mathbb{E}_0^\xi\big((1 \wedge e^{R_i(x,\xi)-\sqrt{2\ell^2/N}\xi_i\zeta_i})\xi_i\big) + \frac{\omega_0(i)}{\lambda_i\sqrt{2\ell^2 N}}.$$

By (2.32) and (2.40),

$$(5.17) \qquad \left| Q(x,\xi) - R_i(x,\xi) + \sqrt{\frac{2\ell^2}{N}}\zeta_i\xi_i \right|^2 \le \frac{M}{N^2}(|\xi_i|^4 + \|C^{1/2}\xi\|_s^4).$$

Noticing that the map $y \mapsto 1 \wedge e^y$ is Lipschitz, we obtain

$$\begin{aligned}
|\omega_0(i)| &\le M\lambda_i\sqrt{N}\,\mathbb{E}_0^\xi\big|\big((1 \wedge e^{Q(x,\xi)}) - (1 \wedge e^{R_i(x,\xi)-\sqrt{2\ell^2/N}\xi_i\zeta_i})\big)\xi_i\big| \\
&\le M\lambda_i\sqrt{N}\left[\mathbb{E}_0^\xi\left| Q(x,\xi) - R_i(x,\xi) + \sqrt{\frac{2\ell^2}{N}}\xi_i\zeta_i \right|^2\right]^{1/2} [\mathbb{E}_0^\xi(\xi_i)^2]^{1/2} \\
&\le \frac{M}{\sqrt{N}}\lambda_i,
\end{aligned}$$

where the last inequality follows from (5.17) and the proof is complete. $\square$

The next lemma takes advantage of the fact that $R_i(x,\xi)$ is independent of ξ_i conditional on x. Thus, using the identity (2.36), we obtain the bound for the approximation made in (2.41).

LEMMA 5.6.

$$\begin{aligned}
&\mathbb{E}_0^\xi\big((1 \wedge e^{R_i(x,\xi)-\sqrt{2\ell^2/N}\zeta_i\xi_i})\xi_i\big) \\
(5.18) \qquad &= -\sqrt{\frac{2\ell^2}{N}}\zeta_i\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N}\Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right) + \omega_1(i), \\
&|\omega_1(i)| \le M|\zeta_i|^2\frac{1}{N}e^{(\ell^2/N)\|\zeta\|^2}.
\end{aligned}$$

PROOF.   Applying (2.36) with $a = -\sqrt{\frac{2\ell^2}{N}}\zeta_i$, $z = \xi_i$ and $b = R_i(x,\xi)$, we obtain the identity

$$\begin{aligned}
&\mathbb{E}_0^\xi\big((1 \wedge e^{R_i(x,\xi)-\sqrt{2\ell^2/N}\xi_i\zeta_i})\xi_i\big) \\
(5.19) \\
&= -\sqrt{\frac{2\ell^2}{N}}\zeta_i\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+(\ell^2/N)\zeta_i^2}\Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|} - \sqrt{\frac{2\ell^2}{N}}|\zeta_i|\right).
\end{aligned}$$

Now we observe that

$$\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N}$$

(5.20)
$$= \mathbb{E}_0^{\xi_i^-} \left(e^{-\sqrt{2\ell^2/N}\sum_{j=1,j\neq i}^N \zeta_j\xi_j - (\ell^2/N)\sum_{j=1,j\neq i}^N \xi_j^2 + (\ell^2/N)\zeta_i^2} \right)$$

$$\leq \mathbb{E}_0^{\xi_i^-} \left(e^{-\sqrt{2\ell^2/N}\sum_{j=1,j\neq i}^N \zeta_j\xi_j + (\ell^2/N)\zeta_i^2} \right) = e^{(\ell^2/N)\|\zeta\|^2}.$$

Since Φ is globally Lipschitz, it follows that

$$\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N} \Phi\left(-\frac{R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|} - \sqrt{\frac{2\ell^2}{N}}|\zeta_i| \right)$$

(5.21)
$$= \mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N} \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|} \right) + \omega_1(i),$$

$$|\omega_1(i)| \leq M|\zeta_i|\frac{1}{\sqrt{N}}\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N} \leq M|\zeta_i|\frac{1}{\sqrt{N}}e^{(\ell^2/N)\|\zeta\|^2},$$

where the last estimate follows from (5.20). The lemma follows from (5.19) and (5.20). $\square$

The next few lemmas are technical and give quantitative bounds for the approximations in (2.43) and (2.44).

LEMMA 5.7.

$$\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N} \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|} \right)$$

$$= \mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi)+\ell^2\zeta_i^2/N} 1_{R_i(x,\xi)<0} + \omega_2(i),$$

$$|\omega_2(i)| \leq M e^{(2\ell^2/N)\|\zeta\|^2}(|\zeta_i|+1)\left[\mathbb{E}_0^\xi \frac{1}{(1+|R(x,\xi)|\sqrt{N})^2} \right]^{1/4}.$$

PROOF. We first prove the following lemma needed for the proof.

LEMMA 5.8. *Let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and CDF of the standard normal distribution, respectively. Then we have*:

(1) *for any $x \in \mathbb{R}$, $|\Phi(-x) - 1_{x<0}| = |1 - \Phi(|x|)|$.*
(2) *for any $x > 0$ and $\epsilon \geq 0$, $1 - \Phi(x) \leq \frac{1+\epsilon}{x+\epsilon}$.*

PROOF. For the first claim, notice that if $x > 0$, $|\Phi(-x) - 1_{x<0}| = |\Phi(-x)| = |1 - \Phi(|x|)|$. If $x < 0$, $|\Phi(-x) - 1_{x<0}| = |1 - \Phi(|x|)|$ and the claim follows.

For the second claim,

$$1 - \Phi(x) = \int_x^\infty \phi(u)\,du \leq \int_x^\infty \frac{u+\epsilon}{x+\epsilon}\phi(u)\,du \leq \frac{\phi(x)+\epsilon}{x+\epsilon} \leq \frac{1+\epsilon}{x+\epsilon}$$

since $\int_{-\infty}^\infty \phi(u)\,du = 1$. $\square$

We now proceed to the proof of Lemma 5.7. By Cauchy–Schwarz and an estimate similar to (5.20),

$$|\omega_2(i)| \leq \mathbb{E}_0^{\xi_i^-}\left[e^{R_i(x,\xi)+(\ell^2/N)\zeta_i^2}\left|1_{R_i(x,\xi)<0} - \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right)\right|\right]$$

$$\leq \left[\mathbb{E}_0^{\xi_i^-} e^{2R_i(x,\xi)+(2\ell^2/N)\zeta_i^2}\right]^{1/2}\left[\mathbb{E}_0^{\xi_i^-}\left|1_{R_i(x,\xi)<0} - \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right)\right|^2\right]^{1/2}$$

$$(5.22)$$

$$\leq Me^{(2\ell^2/N)\|\xi\|^2}\mathbb{E}_0^{\xi_i^-}\left[\left|1_{R_i(x,\xi)<0} - \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right)\right|^2\right]^{1/2}$$

$$\leq Me^{(2\ell^2/N)\|\xi\|^2}\left[\mathbb{E}_0^{\xi_i^-}\left|1_{R_i(x,\xi)<0} - \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right)\right|\right]^{1/2},$$

where the last two observations follow from the computation done in (5.20) and the fact that $|1_{R_i(x,\xi)<0} - \Phi(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|})| < 1$.

By applying Lemma 5.8, with $\epsilon = \frac{1}{\sqrt{2}\ell|\zeta_i|}$,

$$\left|1_{R_i(x,\xi)<0} - \Phi\left(\frac{-R_i(x,\xi)}{\sqrt{2\ell^2/N}|\zeta_i|}\right)\right| = 1 - \Phi\left(\frac{|R_i(x,\xi)|}{\sqrt{2\ell^2/N}|\zeta_i|}\right)$$

$$(5.23) \qquad\qquad = 1 - \Phi\left(\frac{|R_i(x,\xi)|\sqrt{N}}{\sqrt{2}\ell|\zeta_i|}\right)$$

$$\leq (1+\sqrt{2}\ell|\zeta_i|)\frac{1}{1+|R_i(x,\xi)|\sqrt{N}}.$$

The right-hand side of the estimate (5.23) depends on i but we need estimates which are independent of i. In the next lemma, we replace $R_i(x,\xi)$ by $R(x,\xi)$ and control the extra error term.

LEMMA 5.9.

$$(5.24)\quad \mathbb{E}_0^{\xi_i^-}\frac{1}{1+|R_i(x,\xi)|\sqrt{N}} \leq M(1+|\zeta_i|)\left[\mathbb{E}_0^{\xi}\frac{1}{(1+|R(x,\xi)|\sqrt{N})^2}\right]^{1/2}.$$

PROOF.   We write

$$\mathbb{E}_0^{\xi_i^-} \frac{1}{1 + |R_i(x,\xi)|\sqrt{N}} = \mathbb{E}_0^{\xi} \frac{1}{1 + |R_i(x,\xi)|\sqrt{N}}$$

(5.25)
$$= \mathbb{E}_0^{\xi} \frac{1}{1 + |R(x,\xi)|\sqrt{N}} + \gamma$$

$$\leq \left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x,\xi)|\sqrt{N})^2} \right]^{1/2} + \gamma,$$

$$|\gamma| \leq \mathbb{E}_0^{\xi} \left| \frac{1}{1 + |R_i(x,\xi)|\sqrt{N}} - \frac{1}{1 + |R(x,\xi)|\sqrt{N}} \right|$$

$$\leq \mathbb{E}_0^{\xi} \frac{\sqrt{2}\ell|\zeta_i||\xi_i| + \ell^2/\sqrt{N}\xi_i^2}{(1 + |R_i(x,\xi)|\sqrt{N})(1 + |R(x,\xi)|\sqrt{N})}$$

(5.26)
$$\leq \mathbb{E}_0^{\xi} \frac{\sqrt{2}\ell|\zeta_i||\xi_i| + \ell^2/\sqrt{N}\xi_i^2}{(1 + |R(x,\xi)|\sqrt{N})}$$

$$\leq M(|\zeta_i| + 1)\left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x,\xi)|\sqrt{N})^2} \right]^{1/2},$$

and the claim follows from (5.25) and (5.26).   □

Now, by applying the estimates obtained in (5.22), (5.23) and (5.24), we obtain

$$|\omega_2(i)| \leq M e^{(2\ell^2/N)\|\zeta\|^2}(|\zeta_i| + 1)\left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x,\xi)|\sqrt{N})^2} \right]^{1/4}$$

and the proof is complete.   □

The error estimate in ω_2 has $R(x,\xi)$ instead of $R_i(x,\xi)$. This bound can be achieved because the terms $R_i(x,\xi)$ for all $i \in \mathbb{N}$ have the same weak limit as $R(x,\xi)$ and thus the additional error term due to the replacement of $R_i(x,\xi)$ by $R(x,\xi)$ in the expression can be controlled uniformly over i for large N.

LEMMA 5.10.

$$\mathbb{E}_0^{\xi_i^-} e^{R_i(x,\xi) + \ell^2 \zeta_i^2/N} 1_{R_i(x,\xi)<0} = \frac{\beta}{2} + \omega_3(i),$$

$$|\omega_3(i)| \leq M \frac{\zeta_i^2}{N} e^{\ell^2 \|\zeta\|^2/N}$$

$$+ M\left(\frac{1 + |\zeta_i|}{\sqrt{N}} + \frac{1}{N^{3/2}} \sum_{j=1}^{N} |\zeta_j|^3 + \left| 1 - \frac{\|\zeta\|^2}{N} \right| \right)^{1/2}.$$

PROOF.   Set $g(y) \stackrel{\text{def}}{=} e^y 1_{y<0}$. We first need to estimate the following:

$$\left|\mathbb{E}_0^\xi\big(g(R_i(x,\xi)) - g(Z_\ell)\big)\right|.$$

Notice that the function $g(\cdot)$ is not Lipschitz and therefore, the Wasserstein bounds obtained earlier cannot be used directly. However, we use the fact that the normal distribution has a density which is bounded above. So by Lemma 5.3, (5.14) and (5.16),

$$\mathrm{KS}(R_i(x,\xi), Z_\ell) \leq 2M\sqrt{\mathrm{Wass}(R_i(x,\xi), Z_\ell)}$$

$$\leq M\left(\frac{1+|\zeta_i|}{\sqrt{N}} + \frac{1}{N^{3/2}}\sum_{j=1}^N |\zeta_j|^3 + \left|1 - \frac{\|\zeta\|^2}{N}\right|\right)^{1/2}.$$

Since g is positive on $(-\infty, 0]$, for a real valued continuous random variable X,

$$\mathbb{E}(g(X)) = \int_{-\infty}^0 g'(t)\big(\mathbb{P}(X > t)\big)\,dt - g(0)\mathbb{P}(X \geq 0).$$

Hence,

$$|\mathbb{E}_0^\xi g(R_i(x,\xi)) - \mathbb{E}g(Z_\ell)| \leq \left|\int_{-\infty}^0 g'(t)\big(\mathbb{P}(R_i(x,\xi) > t) - \mathbb{P}(Z_\ell > t)\big)\,dt\right|$$

$$+ g(0)\big|\mathbb{P}(R_i(x,\xi) \geq 0) - \mathbb{P}(Z_\ell \geq 0)\big|$$

$$\leq \mathrm{KS}(R_i(x,\xi), Z_\ell)\left(\int_{-\infty}^0 g'(t)\,dt + g(0)\right)$$

$$\leq M\,\mathrm{KS}(R_i(x,\xi), Z_\ell).$$

Hence, putting the above calculations together and noticing that $\mathbb{E}(e^{Z_\ell}1_{Z_\ell<0}) = \beta/2$, we have just shown that

$$\left|\mathbb{E}_0^\xi\big(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\big) - \frac{\beta}{2}\right| \leq M\sqrt{\frac{1+|\zeta_i|}{\sqrt{N}} + \frac{1}{N^{3/2}}\sum_{j=1}^N |\zeta_j|^3 + \left|1 - \frac{\|\zeta\|^2}{N}\right|}.$$

Notice that

$$|\omega_3(i)| \leq \big|e^{\ell^2\zeta_i^2/N}\mathbb{E}_0^\xi\big(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\big) - \beta/2\big|$$

$$\leq \big|e^{\ell^2\zeta_i^2/N} - 1\big|\big|\mathbb{E}_0^\xi\big(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\big)\big|$$

$$+ \big|\mathbb{E}_0^\xi\big(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\big) - \beta/2\big|$$

$$\leq M\frac{\zeta_i^2}{N}e^{\ell^2\|\zeta\|^2/N} + \big|\mathbb{E}_0^\xi\big(e^{R_i(x,\xi)}1_{R_i(x,\xi)<0}\big) - \beta/2\big|,$$

where the last bound follows from (5.20), proving the claimed error bound for $\omega_3(i)$. $\square$

For deriving the error bounds on ω_3, we cannot directly apply the Wasserstein bounds obtained in (5.14), because the function $y \mapsto e^y 1_{y<0}$ is not Lipschitz on $\mathbb{R}$. However, using (5.16), the KS distance between $R_i(x, \xi)$ and Z_ℓ is bounded by the square root of the Wasserstein distance. Thus, using the fact that $e^y 1_{y<0}$ is bounded and positive, we bound the expectation in Lemma 5.10 by the KS distance.

Combining all the above estimates, we see that

$$(5.27) \qquad N\mathbb{E}_0^\xi[x_i^1 - x_i] = -\ell^2 \beta \big(P^N x + C\nabla\Psi(P^N x)\big)_i + r_i^N$$

with

$$(5.28) \qquad |r_i^N| \le |\omega_0(i)| + M\lambda_i\big(\sqrt{N}|\omega_1(i)| + |\zeta_i||\omega_2(i)| + |\zeta_i||\omega_3(i)|\big).$$

The following lemma gives the control over r^N and completes the proof of (2.14), Proposition 2.1.

LEMMA 5.11.   *For $s < \kappa - 1/2$,*

$$\lim_{N\to\infty} \mathbb{E}^{\pi^N} \|r^N\|_s^2 = \lim_{N\to\infty} \mathbb{E}^{\pi^N} \sum_{i=1}^N i^{2s}|r_i^N|^2 = 0.$$

PROOF.   By (5.28), we have $|r_i^N| \le |\omega_0(i)| + M\lambda_i(\sqrt{N}|\omega_1(i)| + |\zeta_i||\omega_2(i)| + |\zeta_i||\omega_3(i)|)$. Therefore,

$$(5.29)\qquad
\begin{aligned}
&\mathbb{E}^{\pi^N} \sum_{i=1}^N i^{2s}|r_i^N|^2 \\
&\quad\le M\mathbb{E}^{\pi^N} \sum_{i=1}^N \big(i^{2s}|\omega_0(i)|^2 + i^{2s}\lambda_i^2\big(N\omega_1(i)^2 + \zeta_i^2\omega_2(i)^2 + \zeta_i^2\omega_3(i)^2\big)\big).
\end{aligned}$$

Now we will evaluate each sum of the right-hand side of the above equation and show that they converge to zero.

- Since $\sum_{i=1}^\infty \lambda_i^2 i^{2s} < \infty$,

$$(5.30) \qquad \sum_{i=1}^N \mathbb{E}^{\pi^N} i^{2s}|\omega_0(i)|^2 \le M\frac{1}{N} \sum_{i=1}^N i^{2s}\lambda_i^2 \le M\frac{1}{N} \sum_{i=1}^\infty \lambda_i^2 i^{2s} \to 0.$$

- By Lemmas 5.6 and 5.2,

$$(5.31) \quad N\mathbb{E}^{\pi^N} \sum_{i=1}^N \lambda_i^2 i^{2s}|\omega_1(i)|^2 \le M\frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\pi^N} \lambda_i^2 i^{2s}|\zeta_i|^4 e^{(2\ell^2/N)\|\zeta\|^2} \to 0.$$

- From Lemma 5.7 and Cauchy–Schwarz, we obtain

$$\sum_{i=1}^{N} \mathbb{E}^{\pi^N} \lambda_i^2 i^{2s} |\zeta_i|^2 |\omega_2(i)|^2$$

$$\leq M \left(\mathbb{E}^{\pi^N} \left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x, \xi)|/\sqrt{N})^2} \right] \right)^{1/2}$$

$$\times \sum_{i=1}^{N} (\mathbb{E}^{\pi^N} e^{(8\ell^2/N)\|\zeta\|^2} \lambda_i^4 i^{4s} (|\zeta_i|^8 + 1))^{1/2}.$$

Proceeding similarly as in Lemma 5.2, it follows that

$$\sum_{i=1}^{N} (\mathbb{E}^{\pi^N} e^{(8\ell^2/N)\|\zeta\|^2} \lambda_i^4 i^{4s} (|\zeta_i|^8 + 1))^{1/2}$$

is bounded in N. Since, with $x \overset{\mathcal{D}}{\sim} \pi_0$, $R(x, \xi)$ converges weakly to Z_ℓ as $N \to \infty$, by the bounded convergence theorem we obtain

$$\lim_{N \to \infty} \mathbb{E}^{\pi_0} \left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x, \xi)|/\sqrt{N})^2} \right] = 0$$

and thus, by Lemma 3.5,

$$\lim_{N \to \infty} \mathbb{E}^{\pi^N} \left[\mathbb{E}_0^{\xi} \frac{1}{(1 + |R(x, \xi)|/\sqrt{N})^2} \right] = 0.$$

Therefore, we deduce that

(5.32) $$\lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E}^{\pi^N} |\zeta_i|^2 i^{2s} \lambda_i^2 |\omega_2(i)|^2 = 0.$$

- After some algebra we obtain from Lemma 5.10 that

$$\mathbb{E}^{\pi^N} \sum_{i=1}^{N} \lambda_i^2 i^{2s} |\zeta_i|^2 |\omega_3(i)|^2$$

$$\leq M \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E}^{\pi^N} \lambda_i^2 i^{2s} |\zeta_i|^6 e^{2\ell^2(\|\zeta\|^2/N)} + M \frac{1}{\sqrt{N}} \mathbb{E}^{\pi^N} \sum_{i=1}^{N} \lambda_i^2 i^{2s} \zeta_i^2 (1 + |\zeta_i|)$$

$$+ M \left[\left(\mathbb{E}^{\pi^N} \left(\frac{1}{N^{3/2}} \sum_{j=1}^{N} |\zeta_j|^3 \right)^2 + \mathbb{E}^{\pi^N} \left| 1 - \frac{\|\zeta\|^2}{N} \right|^2 \right)^{1/2} \right]$$

$$\times \sum_{i=1}^{N} (\mathbb{E}^{\pi^N} \lambda_i^4 i^{4s} \zeta_i^4)^{1/2}.$$

Similar to the previous calculations, using Lemma 5.2, it is quite straightforward to verify that each of the four terms above converges to 0. Thus we obtain

$$(5.33) \qquad \lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E}^{\pi^N} \lambda_i^2 i^{2s} |\zeta_i|^2 |\omega_3(i)|^2 = 0.$$

Now the proof of Lemma 5.11 follows from (5.29)–(5.33). □

This completes the proof of Proposition 2.1, equation (2.14).

5.3. *Rigorous estimates for the diffusion coefficient*: *Proof of Proposition* 2.1, *equation* (2.15). Recall that for $1 \le i, j \le N$,

$$N \mathbb{E}_0[(x_i^1 - x_i^0)(x_j^1 - x_j^0)] = 2\ell^2 \mathbb{E}_0^\xi[(C^{1/2}\xi)_i (C^{1/2}\xi)_j (1 \wedge \exp Q(x, \xi))].$$

The following lemma quantifies the approximations made in (2.48) and (2.49).

LEMMA 5.12.

$$\mathbb{E}_0^\xi[(C^{1/2}\xi)_i (C^{1/2}\xi)_j (1 \wedge \exp Q(x, \xi))] = \lambda_i \lambda_j \delta_{ij} \mathbb{E}^{\xi_{ij}^-}[(1 \wedge \exp R_{ij}(x, \xi))] + \theta_{ij},$$

$$\mathbb{E}^{\xi_{ij}^-}[(1 \wedge \exp R_{ij}(x, \xi))] = \beta + \rho_{ij},$$

where the error terms satisfy

$$(5.34) \quad |\theta_{ij}| \le M \lambda_i \lambda_j (1 + |\zeta_i|^2 + |\zeta_j|^2)^{1/2} \frac{1}{\sqrt{N}},$$

$$(5.35) \quad |\rho_{ij}| \le M \left(\frac{1}{\sqrt{N}} (1 + |\zeta_i| + |\zeta_j|) + \frac{1}{N^{3/2}} \sum_{s=1}^{N} |\zeta_s|^3 + \left| 1 - \frac{\|\zeta\|^2}{N} \right| \right).$$

PROOF. We first derive the bound for θ. Indeed,

$$|\theta_{ij}| \le \mathbb{E}_0^\xi [|(C^{1/2}\xi)_i (C^{1/2}\xi)_j ((1 \wedge e^{Q(x,\xi)}) - (1 \wedge e^{R_{ij}(x,\xi)}))|]$$

$$\le M \lambda_i \lambda_j \mathbb{E}_0^\xi [|\xi_i \xi_j ((1 \wedge e^{Q(x,\xi)}) - (1 \wedge e^{R_{ij}(x,\xi)}))|].$$

By the Cauchy–Schwarz inequality,

$$|\theta_{ij}| \le M \lambda_i \lambda_j (\mathbb{E}_0^\xi |(1 \wedge e^{Q(x,\xi)}) - (1 \wedge e^{R_{ij}(x,\xi)})|)^{1/2}$$

$$\le M \lambda_i \lambda_j (\mathbb{E}_0^\xi |Q(x, \xi) - R_{ij}(x, \xi)|^2)^{1/2}.$$

Using the estimate obtained in (5.2),

$$|\theta_{ij}| \le M \lambda_i \lambda_j (1 + |\zeta_i|^2 + |\zeta_j|^2)^{1/2} \frac{1}{\sqrt{N}}$$

verifying (5.34).

Now we turn to verifying the error bound in (5.35). We need to bound

$$\mathbb{E}_0^\xi\big(g(R_{ij}(x,\xi)) - g(Z_\ell)\big),$$

where $g(y) \stackrel{\text{def}}{=} 1 \wedge e^y$. Notice that $\mathbb{E}(g(Z_\ell)) = \beta$. Since $g(\cdot)$ is Lipschitz,

$$(5.36) \qquad \big|\mathbb{E}_0^\xi\big(g(R_{ij}(x,\xi)) - g(Z_\ell)\big)\big| \le M \,\mathrm{Wass}(R_{ij}(x,\xi), Z_\ell).$$

A simple calculation will yield that

$$\mathrm{Wass}(R_{ij}(x,\xi), R(x,\xi)) \le M(|\zeta_i| + |\zeta_j| + 1)\frac{1}{\sqrt{N}}.$$

Therefore, by the triangle inequality and Lemma 5.3,

$$\mathrm{Wass}(R_{ij}(x,\xi), Z_\ell) \le M\left(\frac{1}{\sqrt{N}}(1 + |\zeta_i| + |\zeta_j|) + \frac{1}{N^{3/2}}\sum_{r=1}^N |\zeta_r|^3 + \left|1 - \frac{\|\zeta\|^2}{N}\right|\right).$$

Hence the estimate in (5.34) follows from the observation made in (5.36). □

Putting together all the estimates produces

$$(5.37) \qquad
\begin{aligned}
N\mathbb{E}_0[(x_i^1 - x_i^0)(x_j^1 - x_j^0)] &= 2\ell^2\beta\lambda_i\lambda_j\delta_{ij} + E_{ij}^N \quad \text{and} \\
|E_{ij}^N| &\le M(|\theta_{ij}| + \lambda_i\lambda_j\delta_{ij}|\rho_{ij}|).
\end{aligned}$$

Finally we estimate the error of E_{ij}^N.

LEMMA 5.13.   *We have*

$$\lim_{N\to\infty}\sum_{i=1}^N \mathbb{E}^{\pi^N}|\langle\phi_i, E^N\phi_j\rangle_s| = 0, \qquad \lim_{N\to\infty}\mathbb{E}^{\pi^N}|\langle\phi_i, E^N\phi_j\rangle_s| = 0$$

for any pair of indices i, j.

PROOF.   From (5.37) we obtain that

$$(5.38) \quad \sum_{i=1}^N \mathbb{E}^{\pi^N}|\langle\phi_i, E^N\phi_i\rangle_s| \le M\left(\sum_{i=1}^N \mathbb{E}^{\pi^N} i^{2s}|\theta_{ii}| + \sum_{i=1}^N \lambda_i^2 i^{2s}\mathbb{E}^{\pi^N}|\rho_{ii}|\right),$$

$$\sum_{i=1}^N \mathbb{E}^{\pi^N} i^{2s}|\theta_{ii}| \le M\sum_{i=1}^N \mathbb{E}^{\pi_0}|\theta_{ii}|i^{2s}$$

$$(5.39) \qquad\qquad\qquad \le M\sum_{i=1}^N \mathbb{E}^{\pi_0}\lambda_i^2 i^{2s}(1 + |\zeta_i|^2)^{1/2}\frac{1}{\sqrt{N}}$$

$$\le M\sum_{i=1}^N \mathbb{E}^{\pi_0}\lambda_i^2 i^{2s}(1 + |\zeta_i|)\frac{1}{\sqrt{N}} \to 0$$

due to the fact that $\sum_{i=1}^{\infty} \lambda_i^2 i^{2s} < \infty$ and Lemma 5.2. Now the second term of (5.38),

$$\sum_{i=1}^{N} \lambda_i^2 i^{2s} \mathbb{E}^{\pi^N} |\rho_{ii}|$$

$$\leq M \mathbb{E}^{\pi_0} \sum_{i=1}^{N} \lambda_i^2 i^{2s} \left(\frac{1}{\sqrt{N}} (1 + |\zeta_i|) + \frac{1}{N^{3/2}} \sum_{s=1}^{N} |\zeta_s|^3 + \left| 1 - \frac{\|\zeta\|^2}{N} \right| \right).$$

The first term above goes to zero by (5.39) and the last term converges to zero by the same arguments used in Lemma 5.2. As mentioned in the proof of the estimate for the term ω_3 in Lemma 5.11, the sum $\mathbb{E}^{\pi^N} \frac{1}{N^{3/2}} \sum_{s=1}^{N} |\zeta_s|^3$ goes to zero. Therefore, we have shown that

$$\lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E}^{\pi^N} |\langle \phi_i, E^N \phi_i \rangle_s| = 0,$$

proving the first claim. Finally, from (5.34) it immediately follows that

$$\mathbb{E}^{\pi} |\langle \phi_i, E^N \phi_j \rangle_s| \leq \mathbb{E}^{\pi} i^s j^s |\theta_{ij}| \to 0,$$

proving the second claim as well. □

Therefore, we have shown

$$N \mathbb{E}_0[(x_i^1 - x_i^0)(x_j^1 - x_j^0)] = 2\ell^2 \beta \langle \phi_i, C \phi_j \rangle + E^N,$$

$$\lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E}^{\pi^N} |\langle \phi_i, E^N \phi_i \rangle| = 0.$$

This finishes the proof of Proposition 2.1, equation (2.15).

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