

# Correction to: A note on diffusion limits of chaotic skew-product flows

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## Abstract

This fixes a gap in the averaging argument in our paper: A note on diffusion limits of chaotic skew product flows. *Nonlinearity* (2011) 1361–1367, and moreover shows that the large deviation estimate assumed there is redundant.

Recall from [2] that  $Z^{(\epsilon)}(t) = \int_0^t g(x^{(\epsilon)}(s), y^{(\epsilon)}(s)) ds$  where  $g(x, y) = f(x, y) - F(x)$ . In [2, Section 3], it is argued that  $Z^{(\epsilon)} \rightarrow 0$  in  $L^1(C([0, T], \mathbb{R}^d); \mu)$ , but the proof is incorrect. Specifically, the proof introduces a random variable  $J_n$  (see below) that depends on  $x^{(\epsilon)}(n\epsilon^{3/2})$  and  $y^{(1)}(s)$ , and derives an estimate for  $\mathbb{E}|J_n|$ . This estimate takes into account the randomness of  $y^{(1)}(s)$  but overlooks the randomness of  $x^{(\epsilon)}(n\epsilon^{3/2})$ .

In this note, we correct the argument in [2]. Moreover, in contrast to [2], our proof does not require any large deviation estimates. Hence the weak invariance principle is a sufficient (as well as necessary) condition for the main result in [2].

**Lemma 1**  $Z^{(\epsilon)} \rightarrow 0$  in  $L^1(C[0, T], \mathbb{R}^d); \mu$  as  $\epsilon \rightarrow 0$  for each  $T > 0$ .

**Proof** Following the calculation in [2, Section 3] with  $\delta = \epsilon^{3/2}$ , we obtain

$$\max_{[0, T]} |Z^{(\epsilon)}| = I_1 + I_2 + O(\epsilon^{3/2}) = I_2 + O(\epsilon^{1/2}) = \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} |J_n| + O(\epsilon^{1/2}), \quad (1)$$

where

$$J_n = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(x^{(\epsilon)}(n\epsilon^{3/2}), y^{(1)}(s)) ds.$$

(The intermediate expressions  $I_1$  and  $I_2$  are defined in [2] but the formulas are not required here.)

For  $u \in \mathbb{R}^d$  fixed, we define

$$\tilde{J}_n(u) = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(u, y^{(1)}(s)) ds = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} A_u \circ \phi_s ds, \quad A_u(y) = g(u, y).$$

Note that  $\tilde{J}_n(u) = \tilde{J}_0(u) \circ \phi_{n\epsilon^{-1/2}}$ , and so  $\mathbb{E}|\tilde{J}_n(u)| = \mathbb{E}|\tilde{J}_0(u)|$ . By the ergodic theorem,  $\mathbb{E}|\tilde{J}_0(u)| \rightarrow 0$  as  $\epsilon \rightarrow 0$  for each  $u$ .

Let  $Q > 0$  and write  $Z^{(\epsilon)} = Z_{Q,1}^{(\epsilon)} + Z_{Q,2}^{(\epsilon)}$  where

$$Z_{Q,1}^{(\epsilon)}(t) = Z^{(\epsilon)}(t)1_{B_\epsilon(Q)}, \quad Z_{Q,2}^{(\epsilon)}(t) = Z^{(\epsilon)}(t)1_{B_\epsilon(Q)^c}, \quad B_\epsilon(Q) = \{\max_{[0,T]} |x^{(\epsilon)}| \leq Q\}.$$

For any  $a > 0$ , there exists a finite subset  $S \subset \mathbb{R}^d$  such that  $\text{dist}(x, S) \leq a/(2 \text{Lip } f)$  for any  $x$  with  $|x| \leq Q$ . Then for all  $n \geq 0$ ,  $\epsilon > 0$ ,

$$1_{B_\epsilon(Q)} |J_n| \leq \sum_{u \in S} |\tilde{J}_n(u)| + a.$$

Hence by (1),

$$\begin{aligned} \mathbb{E} \max_{[0,T]} |Z_{Q,1}^{(\epsilon)}| &\leq \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} \sum_{u \in S} \mathbb{E} |\tilde{J}_n(u)| + Ta + O(\epsilon^{1/2}) \\ &= \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} \sum_{u \in S} \mathbb{E} |\tilde{J}_0(u)| + Ta + O(\epsilon^{1/2}) \leq T \sum_{u \in S} \mathbb{E} |\tilde{J}_0(u)| + Ta + O(\epsilon^{1/2}). \end{aligned}$$

Since  $a > 0$  is arbitrary, we obtain for each fixed  $Q$  that  $\max_{[0,T]} |Z_{Q,1}^{(\epsilon)}| \rightarrow 0$  in  $L^1$ , and hence in probability, as  $\epsilon \rightarrow 0$ .

Next, since  $x^{(\epsilon)} - W^{(\epsilon)}$  is bounded on  $[0, T]$ , for  $Q$  sufficiently large

$$\mu\{\max_{[0,T]} |Z_{Q,2}^{(\epsilon)}| > 0\} \leq \mu\{\max_{[0,T]} |x^{(\epsilon)}| \geq Q\} \leq \mu\{\max_{[0,T]} |W^{(\epsilon)}| \geq Q/2\}.$$

Fix  $c > 0$ . Increasing  $Q$  if necessary, we can arrange that  $\mu\{\max_{[0,T]} |\sqrt{\Sigma}W| \geq Q/2\} < c/4$ . By the continuous mapping theorem,  $\max_{[0,T]} |W^{(\epsilon)}| \rightarrow_d \max_{[0,T]} |\sqrt{\Sigma}W|$ . Hence there exists  $\epsilon_0 > 0$  such that  $\mu\{\max_{[0,T]} |W^{(\epsilon)}| \geq Q/2\} < c/2$  for all  $\epsilon \in (0, \epsilon_0)$ . For such  $\epsilon$ ,

$$\mu\{\max_{[0,T]} |Z_{Q,2}^{(\epsilon)}| > 0\} < c/2.$$

Shrinking  $\epsilon_0$  if necessary, we also have that  $\mu\{\max_{[0,T]} |Z_{Q,1}^{(\epsilon)}| > c/2\} < c/2$ . Hence  $\mu\{\max_{[0,T]} |Z^{(\epsilon)}| > c\} < c$ , and so  $\max_{[0,T]} |Z^{(\epsilon)}| \rightarrow 0$  in probability. Finally, since  $|\max_{[0,T]} |Z^{(\epsilon)}||_\infty \leq 2|f|_\infty T$ , it follows from the bounded convergence theorem that  $\lim_{\epsilon \rightarrow 0} \mathbb{E} \max_{t \in [0,T]} |Z^{(\epsilon)}(t)| = 0$  as required.  $\blacksquare$

**Remark 2** The subsequent paper [1] contains the same error (see [1, Appendix A]). The gap is fixed in identical manner to above, and the large deviation assumptions throughout [1] are again unnecessary.

## References

- [1] G. A. Gottwald and I. Melbourne. Homogenization for deterministic maps and multiplicative noise. *Proc. R. Soc. London A* (2013) 20130201.
- [2] I. Melbourne and A. Stuart. A note on diffusion limits of chaotic skew product flows. *Nonlinearity* (2011) 1361–1367.