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An adaptive Euler–Maruyama scheme for SDEs: convergence and stability

H. LAMBA†
Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA

J. C. MATTINGLY‡
Department of Mathematics, Duke University, Durham, NC 27708, USA

AND

A. M. STUART§
Mathematics Institute, Warwick University, Coventry, CV4 7AL, UK

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The understanding of adaptive algorithms for stochastic differential equations (SDEs) is an open area, where many issues related to both convergence and stability (long-time behaviour) of algorithms are unresolved. This paper considers a very simple adaptive algorithm, based on controlling only the drift component of a time step. Both convergence and stability are studied. The primary issue in the convergence analysis is that the adaptive method does not necessarily drive the time steps to zero with the user-input tolerance. This possibility must be quantified and shown to have low probability. The primary issue in the stability analysis is ergodicity. It is assumed that the noise is nondegenerate, so that the diffusion process is elliptic, and the drift is assumed to satisfy a coercivity condition. The SDE is then geometrically ergodic (averages converge to statistical equilibrium exponentially quickly). If the drift is not linearly bounded, then explicit fixed time step approximations, such as the Euler–Maruyama scheme, may fail to be ergodic. In this work, it is shown that the simple adaptive time-stepping strategy cures this problem. In addition to proving ergodicity, an exponential moment bound is also proved, generalizing a result known to hold for the SDE itself.

Keywords: stochastic differential equations; adaptive time discretization; convergence; stability; ergodicity; exponential moment bounds.

1. Introduction

In this paper, we study the numerical solution of the Itô stochastic differential equation (SDE)

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}W(t), \quad x(0) = X, \tag{1.1}$$

by means of an adaptive time-stepping algorithm. Here, $x(t) \in \mathbb{R}^m$ for each t and $W(t)$ is a d-dimensional Brownian motion. Thus, $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}^{m\times d}$. For simplicity, we assume that the initial condition is deterministic. Throughout, $|\cdot|$ is used to denote either the Euclidean vector

†Email: hlamba@gmu.edu
‡Email: jonm@math.duke.edu
§Corresponding author. Email: stuart@maths.warwick.ac.uk

norm or the Frobenius (trace) matrix norm as appropriate. We assume throughout that f and g are in C^2. Further structural assumptions will be made where needed. The basic adaptive mechanism we study is detailed at the start of Section 2. It is a simple adaptive algorithm, prototypical of a whole class of methods for the adaptive integration of SDEs. Our aim is twofold. First, we show convergence, as the user-input tolerance τ tends to zero; this is a nontrivial exercise because the adaptive strategy does not imply that the time steps taken tend to zero with the tolerance everywhere in phase space. Second, we show that the methods have a variety of desirable properties for the long-time integration of ergodic SDEs, including preservation of ergodicity and exponential moment bounds.

The adaptive method controls the time step of a forward Euler drift step, so that it deviates only slightly from a backward Euler step. This not only controls an estimate of the contribution to the time-stepping error from the drift step but also allows the analysis of stability (long time) properties for implicit backward Euler methods to be employed in the explicit adaptive method. Numerical experiments suggest that both the convergence and the stability analyses extend to a number of more sophisticated methods which control different error measures; some of these experiments are reported below.

It is of interest to discuss our work in the context of a sequence of interesting papers which study the optimality of adaptive schemes for SDEs, using various different error measures (Hofmann *et al*., 2000, 2001, 2002; Müller-Gronbach, to appear). For many of these error measures, which are quite natural in practice, the asymptotically optimal adaptive schemes are based solely on the diffusion. This is essential because it is the diffusion term which dominates the (lack of) regularity in paths and this regularity in turn dominates error measures. Why then have we concentrated on methods which adapt only on the drift? The reason for this is that, as mentioned above, such methods are advantageous for long-time integration. In practice, we anticipate that error controls based on both drift and diffusion could combine the advantages of the asymptotically optimal schemes with the enhanced stability/ergodicity of schemes which control based on the drift.

In order to prove a strong mean-square convergence result for this algorithm, it is first necessary to obtain a suitable upper bound on the sequence of time steps used. These bounds mimic those used in the convergence proofs for adaptive ordinary differential equation (ODE) solvers (Stuart, 1997; Lamba & Stuart, 1998; Lamba, 2000) and require that the numerical solution does not enter neighbourhoods of points where the local error estimate vanishes. (Requiring that these neighbourhoods are small excludes some simple drift vector fields, such as constants. In practice, we would anticipate controlling on both the drift and the diffusion, minimizing this issue.) An essential part of the analysis is a proof that the contribution to the mean-square error from paths that violate this condition is suitably small.

Adaptivity is widely used in the solution of ODEs in an attempt to optimize effort expended per unit of accuracy. The adaptation strategy can be viewed heuristically as a fixed time-step algorithm applied to a time rescaled differential equation (Griffiths, 1988) and it is of interest to study convergence of the algorithms as the tolerance employed to control adaptation is reduced to zero (Lamba & Stuart, 1998). However, adaptation also confers stability on algorithms constructed from explicit time integrators, resulting in better qualitative behaviour than for fixed time-step counter-parts. This viewpoint was articulated explicitly in Sanz-Serna (1992) and subsequently pursued in Aves *et al*. (1997), Higham & Stuart (1998) and Stuart & Humphries (1995), e.g. In particular, the reference Stuart & Humphries (1995) studies the effect of time discretization on dissipative structures such as those highlighted in Hale (1988) and Temam (1989). It is shown that certain adaptive strategies have the desirable property of constraining time steps of explicit integrators, so that the resulting solution update differs in a controlled way from an implicit method. Since many implicit methods have desirable stability properties (see Dekker & Verwer, 1984; Stuart & Humphries, 1996, Chapter 5), this viewpoint can be used to facilitate analysis of the stability of adaptive algorithms (Stuart & Humphries, 1995).

In Mattingly *et al*. (2002), SDEs with additive noise and vector fields satisfying the dissipativity structures of Hale (1988) and Temam (1989) are studied. There, and in Roberts & Tweedie (1996) and Talay (1999, 2002), it is shown that explicit time integrators such as Euler–Maruyama may fail to be ergodic even when the underlying SDE is geometrically ergodic. The reason is that the (mean) dissipativity induced by the drift is lost under time discretization. Since this is exactly the issue arising for explicit integration of dissipative ODEs, and since this issue can be resolved in that context by means of adaptation, it is natural to study how such adaptive methods impact the ergodicity of explicit methods for SDEs. In recent years, the numerical solution of SDEs with gradient drift vector fields has been used as the proposal for a Markov chain Monte Carlo method for sampling from a prescribed density, known only up to a multiplicative constant—a technique referred to as Metropolis-adjusted Langevin algorithm (Cassella & Robert, 2002). In this context, it is very desirable that the time discretization inherits ergodicity. The adaptive scheme proposed here is an approach to ensure this. In this sense, our work complements a number of recent papers concerned with constructing approximation schemes which are ergodic in situations where the standard fixed-step Euler–Maruyama scheme fails to be: in Roberts & Tweedie (1996), a Metropolis–Hastings rejection criterion is used to enforce ergodicity; in Hansen (2002) and Stramer & Tweedie (1999), local linearization is used and in Mattingly *et al*. (2002), implicit methods are used. Although the adaptive method that we analyse here is proved to be convergent on finite time intervals, it would also be of interest to extend the work of Talay (1990), concerned with convergence proofs for invariant measure under time discretization, to the adaptive time-step setting considered here.

In Section 2, we introduce the adaptive algorithm, together with some notations. In Section 3, the finite-time convergence result for the adaptive method is stated. The proof is given in Section 4 and proceeds by extending the fixed-step proof given in Higham *et al*. (2002); the extension is nontrivial because the adaptivity does not force time steps to zero with the tolerance in all parts of the phase space. In Section 5, we state the main results of the paper on the stability of the adaptive method. All results are proved under the dissipativity condition

$$\exists \alpha, \beta \in (0, \infty): \langle f(x), x \rangle \leqslant \alpha - \beta |x|^2 \quad \forall x \in \mathbb{R}^m, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product inducing the Euclidean norm, as well as a boundedness and invertibility condition on the diffusion matrix g. The results proven include ergodicity and an exponential moment bound; all mimic known results about the SDE itself under (1.2). Section 6 starts with a number of *a priori* estimates for the adaptive scheme of Section 2 and proceeds to proofs of the stability stated in Section 5. Numerical results studying both convergence and ergodicity are presented in Sections 7–9. Some concluding remarks and generalizations are given in Section 10.

2. Algorithm

The adaptive algorithm for (1.1) is as follows:

$$k_n = \mathcal{G}(x_n, k_{n-1}), \quad k_{-1} = K,$$

$$x_{n+1} = \mathcal{H}(x_n, \Delta_n) + \sqrt{\Delta_n} g(x_n) \eta_{n+1}, \quad x_0 = X,$$

where $\Delta_n = 2^{-k_n} \Delta_{\max}$. Here,

$$\mathcal{H}(x, t) = x + t f(x)$$

and

$$\mathcal{G}(x,l) = \min\{k \in \mathbb{Z}^+ : |f(\mathcal{H}(x, 2^{-k}\Delta_{\max})) - f(x)| \leqslant \tau \text{ and } k \geqslant l-1\}.$$

The random variables $\eta_j \in \mathbb{R}^d$ form an i.i.d. sequence distributed as $\mathcal{N}(0, I)$. The parameter K defines the initial time step and $\tau > 0$ the tolerance. Note that the algorithm defines a Markov chain for (x_n, k_{n-1}) on $\mathbb{R}^d \times \mathbb{Z}^+$.

We may write

$$\begin{aligned} x_n^\star &= x_n + \Delta_n f(x_n), \\ x_{n+1} &= x_n^\star + \sqrt{\Delta_n} g(x_n)\eta_{n+1}. \end{aligned} \tag{2.1}$$

If $K \in \mathbb{Z}^+$, then $k_n \in \mathbb{Z}^+$ and the error control enforces the condition

$$\Delta_n \leqslant \min\{2\Delta_{n-1}, \Delta_{\max}\},$$

where $\Delta_{\max}$ is the fixed maximum time step. Furthermore, we have

$$|f(x_n^\star) - f(x_n)| \leqslant \tau.$$

In the absence of noise, this implies that the difference between an Euler approximation at the next time step and an explicit second-order approximation is of size $\mathrm{O}(\Delta_n \tau)$. In the presence of noise, it imposes a similar restriction on the means. As mentioned in Section 1, in practice we would anticipate combining this drift error control with others tuned to the diffusion.

2.1 *Notation*

The most important notation conceptually is concerned with making relationships between the numerical approximations at discrete steps and the true solution at certain points in time. To do this, we define $\mathcal{F}_n$ to be the sigma-algebra generated by n steps of the Markov chain for (x_n, k_{n-1}). Let

$$t_n = t_{n-1} + \Delta_{n-1}, \quad t_0 = 0,$$

$\delta > 0$ and define the stopping times N_j by $N_0 = 0$ and, for $j \geqslant 1$,

$$N_j = \inf_{n \geqslant 0} \left\{ n : t_n \geqslant \delta + t_{N_{j-1}} \right\}.$$

Where the dependence on δ is important we will write $N_j(\delta)$. It is natural to examine the approximate process at these stopping times since they are spaced approximately at fixed times in the time variable t. Theorem 5.2 in Section 5 shows that these stopping times are almost surely finite, under the dissipativity condition (1.2). Note that

$$\delta^- := \delta \leqslant t_{N_j} - t_{N_{j-1}} \leqslant \delta + \Delta_{\max} := \delta^+.$$

When considering strong convergence results, it is necessary to interpret $\sqrt{\Delta_n}\eta_{n+1}$ in the adaptive algorithm as the Brownian increment $W(t_{n+1}) - W(t_n)$.

We let

$$y_j = x_{N_{j+1}} \quad \text{and} \quad l_j = k_{N_j}.$$

The Markov chain for $\{y_j, l_j\}$ will be important in our study of long-time behaviour and we will prove that it is ergodic. Let $\mathcal{G}_j = \mathcal{F}_{N_j}$, the filtration of events up to the jth stopping time.

It is convenient to define two continuous-time interpolants of the numerical solution. We set

$$X(t) = x_n, \quad t \in [t_n, t_{n+1}), \tag{2.2}$$

$$\overline{X}(t) = X + \int_0^t f(X(s))\mathrm{d}s + \int_0^t g(X(s))\mathrm{d}W(s). \tag{2.3}$$

Hence, for $t \in [t_n, t_{n+1})$

$$\overline{X}(t) = x_n + (t - t_n) f(x_n) + g(x_n)[W(t) - W(t_n)] \tag{2.4}$$

$$= (1 - \alpha_n(t))x_n + \alpha_n(t)x_n^\star + g(x_n)[W(t) - W(t_n)] \tag{2.5}$$

for $\alpha_n(t) = (t - t_n)/(t_{n+1} - t_n) \in [0, 1)$.

It is sometimes important to know the smallest step size beneath which the error control is always satisfied, at a given point x. Hence, we define

$$k^\star(x) = \min\{k \in \mathbb{Z}^+ : |f(\mathcal{H}(x, 2^{-l}\Delta_{\max})) - f(x)| \leqslant \tau \ \forall l \geqslant k\} \quad \text{and} \quad k^\star(B) = \sup_{x \in B} k^\star(x),$$

noting that, by continuity of f, $k^\star(B)$ is finite if B is bounded.

Because of the boundedness of g, we deduce that there are functions $\sigma(x)$ and constants $\sigma, a > 0$ such that, for η distributed as η_1 and independent of x,

$$\mathbb{E}|g(x)\eta|^2 := \sigma^2(x) \leqslant \sigma^2 \quad \text{and} \quad \mathbb{E}||g(x)\eta|^2 - \sigma^2(x)|^2| \leqslant a\sigma^4.$$

The following definitions will be useful:

$$\tilde{\alpha} = \alpha + \frac{1}{2}\tau, \quad \tilde{\beta} = \beta - \frac{1}{2}\tau, \quad \beta_n = \frac{1}{1 + 2\tilde{\beta}\Delta_n}, \quad \bar{\gamma} = 1 + \tilde{\beta}\Delta_{\max}.$$

We will always assume that τ is chosen small enough so that $\tilde{\beta} > 0$. The constant γ^- is chosen so that

$$(1 + t)^{-1} \leqslant (1 - \gamma^- t) \leqslant \mathrm{e}^{-\gamma^- t} \quad \forall t \in [0, 2\tilde{\beta}\Delta_{\max}].$$

3. Convergence result

We start by discussing the error-control mechanism. We define $F_1(u)$ by

$$F_1(u) = \mathrm{d}f(u)f(u),$$

the function $F_2(u, h)$ by

$$F_2(u, h) := h^{-1}(f(u + hf(u)) - f(u) - hF_1(u))$$

and $E(u, h)$ by

$$E(u, h) = f(u + hf(u)) - f(u).$$

Now, since $f \in C^2$, Taylor series expansion gives

$$E(u, h) = h[F_1(u) + hF_2(u, h)], \tag{3.1}$$

where F_1 and F_2 are as defined above. Note that the error control forces a time step so that the norm of $E(x_n, \Delta_n)$ is of order $\mathrm{O}(\tau)$. Estimating the implications of this for the time step Δ_n forms the heart of the convergence proof below.

In order to state the assumptions required for the convergence result, we define, for $R, \epsilon \geqslant 0$, the sets

$$\Psi(\epsilon) = \{u \in \mathbb{R}^m : |F_1(u)| \leqslant \epsilon\}, \quad B_R = \{u \in \mathbb{R}^m : |u| \leqslant R\} \quad \text{and} \quad B_{R,\epsilon} = B_R \backslash \Psi(\epsilon)$$

and introduce the constant $K_R = \sup_{u \in B_R, h \in [0, \Delta t_{\max}]} |F_2(u, h)|$. Now define the following:

$$\begin{aligned}
\sigma_R &:= \inf\{t \geqslant 0 : |\overline{X}(t)| \geqslant R\}, & \rho_R &:= \inf\{t \geqslant 0 : |x(t)| \geqslant R\}, & \theta_R &:= \sigma_R \wedge \rho_R, \\
\sigma_\epsilon &:= \inf\{t \geqslant 0 : |F_1(\overline{X}(t))| \leqslant \epsilon\}, & \rho_\epsilon &:= \inf\{t \geqslant 0 : |F_1(x(t))| \leqslant 2\epsilon\}, & \theta_\epsilon &:= \sigma_\epsilon \wedge \rho_\epsilon, \\
\sigma_{R,\epsilon} &:= \sigma_R \wedge \sigma_\epsilon, & \rho_{R,\epsilon} &:= \rho_R \wedge \rho_\epsilon, & \theta_{R,\epsilon} &:= \theta_R \wedge \theta_\epsilon.
\end{aligned}$$

The first assumption is a local Lipschitz condition on the drift and diffusion coefficients, together with moment bounds on the true and numerical solutions.

ASSUMPTION 3.1 For each $R > 0$, there exists a constant C_R, depending only on R, such that

$$|f(a) - f(b)|^2 \vee |g(a) - g(b)|^2 \leqslant C_R |a - b|^2 \quad \forall a, b \in \mathbb{R}^m \quad \text{with } |a| \vee |b| \leqslant R. \tag{3.2}$$

For some $p > 2$, there is a constant A, uniform in $\tau \to 0$, such that

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |\overline{X}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |x(t)|^p\right] \leqslant A. \tag{3.3}$$

Note that inequality (3.2) is a local Lipschitz assumption which will be satisfied for any f and g in C^2. The inequality (3.3) states that the pth moments of the exact and numerical solution are bounded for some $p > 2$. Theorem 5.4 proves (3.3) for the numerical interpolant, under natural assumption on f and g (see Assumption 5.1). Under the same assumption, such a bound is known to hold for $x(t)$; see Mao (1997).

We clearly also need an assumption on the local error estimate since if, e.g. the drift term $f(u)$ were constant, then $E(u, h) \equiv 0$ and the step-size would, through doubling, reach $\Delta_{\max}$, no matter how small τ is, and convergence cannot occur as $\tau \to 0$. Because the function $F_1(u)$ maps $\mathbb{R}^m$ into itself, the following assumption on the zeros of $F_1(u)$ will hold for generic drift functions f which are nonconstant on any open set; it does exclude, however, the case of constant drift. Furthermore, the assumption on the hitting time rules out dimension $m = 1$.

ASSUMPTION 3.2 Define

$$\ell(\epsilon, R) = d_{\mathrm{H}}\{\Psi(2\epsilon)^c \cap B_R, \Psi(\epsilon) \cap B_R\}.$$

For any given $R > 0$, we assume that $\ell(\epsilon, R) > 0$ for all sufficiently small $\epsilon > 0$, and that $\ell(\epsilon, R) \to 0$ as $\epsilon \to 0$. Furthermore, the hitting time ρ_ϵ satisfies, for any $X \notin \Psi(0)$,

$$\rho_\epsilon \to \infty \quad \text{as } \epsilon \to 0 \text{ a.s.}$$

Here, d_{H} denotes the Hausdorff distance. The preceding assumption requires that the contours defining the boundary of $\Psi(\epsilon)$ are strictly nested as ϵ increases and bounded. This enables us to show that the probability of $(x(t), \overline{X}(t)) \in (\Psi(2\epsilon)^c \cap B_R) \times (\Psi(\epsilon) \cap B_R)$ is small, a key ingredient in the proof.

We now state the strong convergence of the adaptive numerical method, using the continuous-time interpolant $\overline{X}(t)$. Note that we do not assume $\Delta_{\max} \to 0$ for this theorem. Hence, the nonstandard part of the proof comes from estimating the contribution to the error from regions of phase space where the time step is not necessarily small as $\tau \to 0$.

THEOREM 3.3 Assume that $X \notin \Psi(0)$. Let Assumptions 3.1 and 3.2 hold. Then, there is $\Delta_c(\tau)$ such that, for all $\Delta_{-1} < \Delta_c(\tau)$ and any $T > 0$, the numerical solution with continuous-time extension $\overline{X}(t)$ satisfies

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |\overline{X}(t) - x(t)|^2\right] \to 0 \quad \text{as } \tau \to 0.$$

4. Proof of convergence result

The primary technical difficulty to address in convergence proofs for adaptive methods is to relate the time step to the tolerance τ. Roughly speaking, the formula (3.1) shows that, provided $F_1(u) \neq 0$, the error control will imply $\Delta_n = \mathrm{O}(\tau)$. We now make this precise. We provide an upper bound on the time step sequence of numerical solutions that remain within $B_{R,\epsilon}$, for sufficiently small τ. For given $R, \epsilon > 0$, we define the quantities

$$\overline{h}_{R,\epsilon} = \frac{\epsilon}{6K_R} \quad \text{and} \quad \tau_{R,\epsilon} = \frac{\epsilon^2}{12K_R}.$$

LEMMA 4.1 For any $R, \epsilon > 0$, if $\{x_n\}_{n=0}^N \subseteq B_{R,\epsilon}$, $\tau < \tau_{R,\epsilon}$ and $\Delta_{-1} < \frac{2\tau}{\epsilon}$, then

$$\Delta_n \leqslant \min\left\{\overline{h}_{R,\epsilon}, \frac{2\tau}{\epsilon}\right\} \quad \forall n : 0 \leqslant n \leqslant N. \tag{4.1}$$

Proof. The error control implies

$$|E(x_n, \Delta_n)| = \Delta_n |F_1(x_n) + \Delta_n F_2(x_n, \Delta_n)| \leqslant \tau.$$

Note that

$$\Delta_{-1} < 2\tau_{R,\epsilon}/\epsilon = \overline{h}_{R,\epsilon}.$$

We first proceed by contradiction to prove $\Delta_n \leqslant \overline{h}_{R,\epsilon}\ \forall n : 0 \leqslant n \leqslant N$. Let $0 \leqslant m \leqslant N$ be the first integer such that $\Delta_m > \overline{h}_{R,\epsilon}$. Then, since there is a maximum time step ratio of 2, we have

$$\Delta_m \in \left(\frac{\epsilon}{6K_R}, \frac{\epsilon}{3K_R}\right] \Rightarrow \Delta_m |F_2(x_m, \Delta_m)| < \frac{\epsilon}{2}$$

$$\Rightarrow |E(x_m, \Delta_m)| > \Delta_m(\epsilon - \epsilon/2) \geqslant \frac{\epsilon \overline{h}_{R,\epsilon}}{2} = \frac{\epsilon^2}{12K_R} = \tau_{R,\epsilon} > \tau.$$

Thus, Δ_m is not an acceptable time step, contradicting our original assumption and hence the first result follows. The proof of the bound on the time step in (4.1) now follows immediately since

$$\Delta_n \leqslant \frac{\tau}{|F_1(x_n) + \Delta_n F_2(x_n, \Delta_n)|} \leqslant \frac{\tau}{(\epsilon - \epsilon/2)} \leqslant \frac{2\tau}{\epsilon} \quad \forall n : 0 \leqslant n \leqslant N.$$

□

Proof of Theorem 3.3. We denote the error by

$$e(t) := \overline{X}(t) - x(t).$$

Recall the Young's inequality: for $r^{-1} + q^{-1} = 1$,

$$ab \leqslant \frac{\delta}{r} a^r + \frac{1}{q\delta^{q/r}} b^q \quad \forall a, b, \delta > 0.$$

We thus have for any $\delta > 0$,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^2\right] &= \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^2 \mathbf{1}\{\theta_{R,\epsilon} > T\}\right] + \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^2 \mathbf{1}\{\theta_{R,\epsilon} \leqslant T\}\right] \\ &\leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t \wedge \theta_{R,\epsilon})|^2 \mathbf{1}\{\theta_{R,\epsilon} > T\}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^p\right] \\ &\quad + \frac{1 - \frac{2}{p}}{\delta^{2/(p-2)}}\mathbb{P}(\theta_{R,\epsilon} \leqslant T). \end{aligned} \tag{4.2}$$

Now

$$\mathbb{P}(\theta_{R,\epsilon} \leqslant T) = \mathbb{P}\{\theta_R \leqslant T\} + \mathbb{P}\{\theta_\epsilon \leqslant T, \theta_R > T\}.$$

But

$$\mathbb{P}\{\theta_R \leqslant T\} \leqslant \mathbb{P}\{\sigma_R \leqslant T\} + \mathbb{P}\{\rho_R \leqslant T\},$$

while

$$\mathbb{P}\{\theta_\epsilon \leqslant T, \theta_R > T\} \leqslant \mathbb{P}\{\rho_\epsilon \leqslant T\} + \mathbb{P}\{\theta_\epsilon \leqslant T, \theta_R > T, \rho_\epsilon > T\}.$$

Thus, we have

$$\mathbb{P}(\theta_{R,\epsilon} \leqslant T) \leqslant \mathbb{P}(\sigma_R \leqslant T) + \mathbb{P}(\rho_R \leqslant T) + \mathbb{P}(\rho_\epsilon \leqslant T) + \mathbb{P}\{\theta_\epsilon \leqslant T, \theta_R > T, \rho_\epsilon > T\}.$$

To control the last term, note that whenever $\theta_\epsilon \leqslant T$, $\theta_R > T$ and $\rho_\epsilon > T$, we know that $|e(\sigma_\epsilon)| \geqslant \ell(\epsilon, R)$. Hence, we have

$$\mathbb{P}\{\theta_\epsilon \leqslant T, \theta_R > T, \rho_\epsilon > T\} \leqslant \mathbb{P}\{|e(T \wedge \theta_{R,\epsilon})| \geqslant \ell(\epsilon, R)\} \leqslant \mathbb{E}|e(T \wedge \theta_{R,\epsilon})|^2 / \ell(\epsilon, R)^2.$$

Combining the two preceding inequalities gives

$$\mathbb{P}(\theta_{R,\epsilon} \leqslant T) \leqslant \mathbb{P}(\sigma_R \leqslant T) + \mathbb{P}(\rho_R \leqslant T) + \mathbb{P}(\rho_\epsilon \leqslant T) + \mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |e(t \wedge \theta_{R,\epsilon})|^2\right) \Big/ \ell(\epsilon, R)^2.$$

By Markov's inequality,

$$\mathbb{P}\{\sigma_R \leqslant T\}, \mathbb{P}\{\rho_R \leqslant T\} \leqslant \frac{A}{R^p}$$

so that

$$\mathbb{P}(\theta_{R,\epsilon} \leqslant T) \leqslant \frac{2A}{R^p} + \mathbb{P}(\rho_\epsilon \leqslant T) + \mathbb{E}\left(\sup_{0\leqslant T} |e(t \wedge \theta_{R,\epsilon})|^2\right) \Big/ \ell(\epsilon, R)^2. \tag{4.3}$$

Furthermore,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^p\right] \leqslant 2^{p-1}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} (|\overline{X}(t)|^p + |x(t)|^p)\right] \leqslant 2^p A. \tag{4.4}$$

Using (4.3) and (4.4) in (4.2) gives, for ϵ sufficiently small,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t)|^2\right] \leqslant \left(1 + \frac{p-2}{p\delta^{2/(p-2)}\ell(\epsilon, R)^2}\right)\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |e(t \wedge \theta_{R,\epsilon})|^2\right] + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)}{p\delta^{2/(p-2)}}\left[\frac{2A}{R^p} + \mathbb{P}\{\rho_\epsilon \leqslant T\}\right]. \tag{4.5}$$

Take any $\kappa > 0$. To complete the proof, we choose δ sufficiently small so that the second term on the right-hand side of (4.5) is bounded by $\kappa/4$ and R and ϵ sufficiently large/small so that the third and fourth terms are bounded by $\kappa/4$. Now reduce τ so that Lemma 4.1 applies. Then, by further reduction of τ in Lemma 4.2, we upper bound the first term by $\kappa/4$. (Lemma 4.2 calculates the error conditioned on the true and numerical solutions staying within a ball of radius R, and away from small sets where the error-control mechanism breaks down. With this conditioning, it follows from Lemma 4.1 that we have $\Delta_n = \mathrm{O}(\tau)$, which is the essence of why Lemma 4.2 holds.)

Consequently, we have

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\overline{X}(t) - x(t)|^2\right] \leqslant \kappa$$

and since κ is arbitrary the required result follows. □

In the following, C is a universal constant independent of T, R, ϵ, δ and τ. Likewise, C_R is a universal constant depending upon R, but independent of T, ϵ, δ and τ, $C_{R,T}$ is a universal constant depending upon R and T, but independent of ϵ, δ and τ and $C_{R,\epsilon,T}$ and so forth are defined similarly. The actual values of these constants may change from one occurrence to the next.

LEMMA 4.2 Assume that $X \notin \Psi(0)$ and that τ is sufficiently small for the conditions of Lemma 4.1 to hold. Then, the continuous interpolant of the numerical method, $\overline{X}(t)$, satisfies the following error bound:

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\overline{X}(t \wedge \theta_{R,\epsilon}) - x(t \wedge \theta_{R,\epsilon})|^2\right] \leqslant C_{R,\epsilon,T}\tau.$$

Proof. Using

$$x(t\wedge\theta_{R,\epsilon}) := X + \int_0^{t\wedge\theta_{R,\epsilon}} f(x(s))\mathrm{d}s + \int_0^{t\wedge\theta_{R,\epsilon}} g(x(s))\mathrm{d}W(s),$$

(2.3) and the Cauchy–Schwartz inequality, we have that $\chi := |\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})|^2$ satisfies

$$\begin{aligned}\chi &= \left|\int_0^{t\wedge\theta_{R,\epsilon}} (f(X(s)) - f(x(s)))\mathrm{d}s + \int_0^{t\wedge\theta_{R,\epsilon}} (g(X(s)) - g(x(s)))\mathrm{d}W(s)\right|^2\\ &\leqslant 2\left[T\int_0^{t\wedge\theta_{R,\epsilon}} |f(X(s)) - f(x(s))|^2\,\mathrm{d}s + \left|\int_0^{t\wedge\theta_{R,\epsilon}} (g(X(s)) - g(x(s)))\mathrm{d}W(s)\right|^2\right].\end{aligned}$$

Let

$$E(s) := \left[\sup_{0\leqslant t\leqslant s} |\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})|^2\right].$$

Then, from the local Lipschitz condition (3.2) and the Doob–Kolmogorov martingale inequality (Rogers & Williams, 2000), we have for any $t^* \leqslant T$

$$\begin{aligned}\mathbb{E}E(t^*) &\leqslant 2C_R(T+4)\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s) - x(s)|^2\,\mathrm{d}s\\ &\leqslant 4C_R(T+4)\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} [|X(s) - \overline{X}(s)|^2 + |\overline{X}(s) - x(s)|^2]\mathrm{d}s\\ &\leqslant 4C_R(T+4)\left[\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s) - \overline{X}(s)|^2\,\mathrm{d}s + \int_0^{t^*}\mathbb{E}E(s)\mathrm{d}s\right]. \qquad (4.6)\end{aligned}$$

Given $s \in [0, T\wedge\theta_{R,\epsilon})$, let k_s be the integer for which $s \in [t_{k_s}, t_{k_s+1})$. Note that t_{k_s} is a stopping time because Δ_{k_s} is a deterministic function of $(x_{k_s}, \Delta_{k_s-1})$. We now bound the right-hand side in (4.6). From the local Lipschitz condition (3.2), a straightforward calculation shows that

$$|X(s) - \overline{X}(s)|^2 \leqslant C_R\left(|x_{k_s}|^2 + 1\right)\left(\Delta_{k_s}^2 + |W(s) - W(t_{k_s})|^2\right).$$

Now, for $s < \theta_{R,\epsilon}$, using Lemma 4.1,

$$\begin{aligned}|W(s) - W(t_{k_s})|^2 &= s - t_{k_s} + 2\int_{t_{k_s}}^s [W(l) - W(t_{k_s})]\,\mathrm{d}W(l)\\ &\leqslant (s - t_{k_s})[1 + I(s)] \leqslant \frac{2\tau}{\epsilon}[1 + I(s)].\end{aligned}$$

Here,

$$I(s) = \frac{2}{(s - t_{k_s})}\left|\int_{t_{k_s}}^s [W(l) - W(t_{k_s})]\,\mathrm{d}W(l)\right|.$$

Let $\mathcal{H}_s$ denote the σ-algebra of Brownian paths up to time t_{k_s}. Then, conditioned on $\mathcal{H}_s$, we have

$$\mathbb{E}I(s) \leqslant \sqrt{2}. \tag{4.7}$$

Thus, using Lemma 4.1, (3.3) and the Lyapunov inequality (Kloeden & Platen, 1991),

$$\begin{aligned}\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s)-\overline{X}(s)|^2\,\mathrm{d}s &\leqslant \mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} C_R\left(|x_{k_s}|^2+1\right)\left(4\tau^2/\epsilon^2+\left|W(s)-W\left(t_{k_s}\right)\right|^2\right)\mathrm{d}s\\ &\leqslant C_{R,\epsilon}\tau\mathbb{E}\int_0^{t^*}\left(1+|x_{k_s}|^2\right)(1+I(s))\mathrm{d}s\\ &\leqslant C_{R,\epsilon,T}(A^{2/p+1})\tau.\end{aligned}$$

To obtain the last line, we condition on $\mathcal{H}_s$ so that $|x_{k_s}|^2$ and $I(s)$ are independent; we then use (4.7) and the assumed moment bound.

In (4.6), we then have by Lemma 4.1,

$$\mathbb{E}E(t^*) \leqslant C_{R,\epsilon,T}\tau + 4C_{R,T}\int_0^{t^*}\mathbb{E}E(s)\mathrm{d}s.$$

Applying the Gronwall inequality, we obtain

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}(\overline{X}(t\wedge\theta_{R,\epsilon})-x(t\wedge\theta_{R,\epsilon}))^2\right] \leqslant C(R,\epsilon,T)\tau.$$

□

5. Stability results

For all our stability results, in this and the following sections, we make the assumption that (1.2) holds, together with some conditions on the diffusion matrix. To be explicit, we make the following assumption:

ASSUMPTION 5.1 There exists finite positive α, β such that

$$\langle f(x), x\rangle \leqslant \alpha - \beta|x|^2 \quad \forall\, x \in \mathbb{R}^m,$$

where $\langle\cdot,\cdot\rangle$ is the inner product inducing the Euclidean norm $|\cdot|$. Furthermore, $m = d$ and g is globally bounded and globally invertible.

The assumption is made, without explicit statement, for the remainder of the paper. We also assume, without explicit statement, that $\tau < 2\beta$ so that $\tilde{\beta} > 0$. Finally, we assume, also without explicit statement, that there is at least one point $\overline{y} \in \mathbb{R}^m$ such that

$$k^*(\overline{y}) = \mathcal{G}(\overline{y}, 1). \tag{5.1}$$

This may implicitly force upper bounds on τ and $\Delta_{\max}$, although neither is necessarily restricted by this assumption. The existence of such a $\overline{y}$ is implied by Assumption 5.1, which rules out f being identically constant. Then there exists $\overline{y}$ for which the function

$$|f(\overline{y}+hf(\overline{y}))-f(\overline{y})|$$

is nonzero in a neighbourhood of $h = 0$ and (5.1) must hold, possibly after enforcing bounds on τ and $\Delta_{\max}$.

Under Assumption 5.1, the solution of (1.1) exists for all $t > 0$ (Has'minskii, 1980; Mao, 1997) and the equation is geometrically ergodic (Has'minskii, 1980; Meyn & Tweedie, 1992; Mattingly *et al.*, 2002). The first stability result ensures that the method will not decrease its step-size in such a way that it is unable to reach arbitrary finite times.

THEOREM 5.2 The stopping times N_j are almost surely finite.

The next result is the main ergodic result of the paper. It ensures that the adaptive method has an attracting statistical steady state. Letting $\mathbb{E}^{y,l}$ denote the expectation under the Markov chain started at $x_0 = y$ and $k_{-1} = l$, we have the following result. (Recall δ occurring in the definition of stopping times N_j.)

THEOREM 5.3 Assume that $\delta > 5\Delta_{\max}$. The Markov chain $\{y_j, l_j\} = \{x_{N_j+1}, k_{N_j}\}$ has a unique invariant measure π. Furthermore, if $h\colon \mathbb{R}^m \times \mathbb{Z}^+ \to \mathbb{R}$ is measurable and

$$|h(y,l)| \leqslant 1 + |y|^2 \quad \forall\, (y,l) \in \mathbb{R}^m \times \mathbb{Z}^+,$$

then there exists $\lambda \in (0,1), \kappa \in (0,\infty)$ such that

$$|\mathbb{E}^{y_0,l_0} h(y_n, l_n) - \pi(h)| \leqslant \kappa \lambda^n [1 + |y_0|^2].$$

The final result gives a moment bound on the continuous-time interpolants of the numerical solution, mimicking that for the SDE itself.

THEOREM 5.4 There exists a $\lambda > 0$ and a $c > 0$ so that

$$\mathbb{E} \exp\left(\lambda \sup_{t\in[0,T]} \|X(t)\|^2 \right) \leqslant \exp(\lambda |X|^2 + cT),$$

$$\mathbb{E} \exp\left(\lambda \sup_{t\in[0,T]} \|\overline{X}(t)\|^2 \right) \leqslant \exp(\lambda |X|^2 + cT).$$

6. Proof of stability results

We start with a number of estimates which will be needed to prove the main results. It is useful to define

$$\xi_{n+1} = 2\sqrt{\Delta_n}\langle x_n^\star, g(x_n)\eta_{n+1}\rangle, \quad \tilde{\xi}_{n+1} = \Delta_n[|g(x_n)\eta_{n+1}|^2 - \sigma^2(x_n)],$$

$$M_n = \sum_{j=0}^{n-1} \xi_{j+1}, \quad \tilde{M}_n = \sum_{j=0}^{n-1} \tilde{\xi}_{j+1}.$$

Observe that $\langle x_n^\star, g(x_n)\eta_{n+1}\rangle$ is a Gaussian random variable conditioned on the values of x_n and $x_n^\star$. Hence, the last two expressions are Martingales satisfying the assumptions of Lemma A.1 from the appendix. Also note that the quadratic variations satisfy

$$\langle M\rangle_n \leqslant \sum_{j=0}^{n-1} 4\Delta_j |x_j^\star|^2 \sigma^2 \quad \text{and} \quad \langle \tilde{M}\rangle_n \leqslant \sum_{j=0}^{n-1} a\Delta_j^2 \sigma^4. \tag{6.1}$$

We start with a straightforward lemma.

LEMMA 6.1 The sequences $\{x_n^\star\}$ and $\{x_n\}$ satisfy

$$|x_n^\star|^2 \leqslant |x_n|^2 + 2\Delta_n[\tilde{\alpha} - \tilde{\beta}|x_n^\star|^2],$$
$$|x_{n+1}|^2 \leqslant \beta_n|x_n|^2 + \Delta_n[2\tilde{\alpha} + \sigma^2] + \xi_{n+1} + \tilde{\xi}_{n+1}.$$

Hence,

$$\langle M\rangle_n \leqslant 4\sigma^2 \sum_{j=0}^{n-1} |x_j|^2 \Delta_j + 8\sigma^2\tilde{\alpha} \sum_{j=0}^{n-1} \Delta_j^2.$$

Proof. Taking the inner product of the equation

$$x_n^\star = x_n + \Delta_n f(x_n),$$

with $x_n^\star$ and using the fact that the error control implies

$$|f(x_n^\star) - f(x_n)| \leqslant \tau,$$

a straightforward calculation from Stuart & Humphries (1995), using (1.2), gives the first result. To get the second, simply square the expression (2.1) for x_{n+1} and use the first, noting that $\beta_n \leqslant 1$. For the third, use the first in the bound (6.1) for $\langle M\rangle_n$. □

LEMMA 6.2 We have

$$|x_{n+1}|^2 \leqslant |X|^2 + C_0 t_{n+1} + M_{n+1} - \frac{1}{2}\frac{\tilde{\beta}}{\sigma^2}\langle M\rangle_{n+1} + \tilde{M}_{n+1} - 2\langle\tilde{M}\rangle_{n+1},$$

where $C_0 = [2\tilde{\alpha} + 4\sigma^4\Delta_{\max}]$. Furthermore,

$$\mathbb{P}\left(\sup_{0\leqslant n}\{|x_n|^2 - C_0 t_n\} \geqslant |X|^2 + A\right) \leqslant 2\exp(-BA),$$

where B is a positive constant depending only on σ and $\tilde{\beta}$.

Proof. Squaring the expression for x_{n+1} in (2.1), bounding $|x_n^\star|^2$ by the first inequality in Lemma 6.1 and summing gives

$$|x_{n+1}|^2 \leqslant |X|^2 + C_0 t_{n+1} + S_{n+1} + \tilde{S}_{n+1},$$

where

$$S_{n+1} = M_{n+1} - 2\tilde{\beta}\sum_{k=0}^{n} |x_k^\star|^2\Delta_k, \quad \tilde{S}_{n+1} = \tilde{M}_{n+1} - 4\sigma^4\Delta_{\max}t_{n+1}$$

and M_n, $\tilde{M}_n$ are as before. Using (6.1), one obtains

$$\langle \tilde{M} \rangle_{n+1} \leqslant 2\sigma^4 \Delta_{\max} t_{n+1}$$

and

$$\langle M \rangle_{n+1} \leqslant 4\sigma^2 \sum_{k=0}^{n} \Delta_k |x_k^\star|^2 \, .$$

Combining all these produces

$$S_{n+1} \leqslant M_{n+1} - \frac{1}{2}\frac{\tilde{\beta}}{\sigma^2}\langle M \rangle_{n+1} \quad \text{and} \quad \tilde{S}_{n+1} \leqslant \tilde{M}_{n+1} - 2\langle \tilde{M} \rangle_{n+1} \, .$$

The probabilistic estimate follows from the exponential martingale estimates from the appendix. □

COROLLARY 6.3 There exists a universal $\lambda > 0$ and $C_1 > 0$, so that for any stopping time N with $0 \leqslant t_N \leqslant t_*$ almost surely, for some fixed number t_*, one has

$$\mathbb{E} \exp\left(\lambda \sup_{0 \leqslant n \leqslant N} |x_n|^2 \right) \leqslant C_1 \exp(\lambda |X|^2 + \lambda C_0 t_*).$$

Proof. The result follows from Lemma 6.2 and the observation that

$$\mathbb{P}\left(\sup_{0 \leqslant n \leqslant N} |x_n|^2 \geqslant |X|^2 + C_0 t_* + A \right) \leqslant \mathbb{P}\left(\sup_{0 \leqslant n} \{ |x_n|^2 - C_0 t_n \} \geqslant |X|^2 + A \right).$$

□

LEMMA 6.4 The Markov chain $\{x_{N_j}\}_{j \in \mathbb{Z}^+}$ satisfies the Foster–Lyapunov drift condition

$$\mathbb{E}\left\{ \left| x_{N_{j+1}} \right|^2 \left| \mathcal{F}_{N_j} \right. \right\} \leqslant \exp(-2\gamma^- \tilde{\beta} \delta^-) \left| x_{N_j} \right|^2 + \exp(2\tilde{\beta}\delta^+)[2\tilde{\alpha} + \sigma^2]\delta^+.$$

That is

$$\mathbb{E}\{|y_{j+1}|^2 | \mathcal{G}_j\} \leqslant \exp(-2\gamma^- \tilde{\beta}\delta^-)|y_j|^2 + \exp(2\tilde{\beta}\delta^+)[2\tilde{\alpha} + \sigma^2]\delta^+.$$

Proof. Note that $(1+x)^{-1} \leqslant \mathrm{e}^{-\gamma^- x}$ for all $x \in [0, 2\Delta_{\max}\tilde{\beta}]$. From Lemma 6.1, we have

$$|x_{n+1}|^2 \leqslant \beta_n |x_n|^2 + \kappa_n + \xi_{n+1} + \tilde{\xi}_{n+1},$$

where $\kappa_n := \Delta_n[2\tilde{\alpha} + \sigma^2]$. Defining

$$\gamma_j = \left(\prod_{l=0}^{j-1} \beta_l^{-1} \right),$$

we obtain

$$\mathbb{E}\left(\gamma_{N_{j+1}}\left|x_{N_j+1}\right|^2|\mathcal{F}_{N_j}\right) \leqslant \gamma_{N_j}\left|x_{N_j}\right|^2 + \mathbb{E}\left(\sum_{l=N_j}^{N_{j+1}-1}\gamma_{l+1}\kappa_l\big|\mathcal{F}_{N_j}\right).$$

Now

$$\sum_{l=N_j}^{N_{j+1}-1}\Delta_j \leqslant \delta + \Delta_{\max} = \delta^+ \quad \text{and} \quad \sum_{l=N_j}^{N_{j+1}-1}\Delta_j \geqslant \delta = \delta^-. \tag{6.2}$$

Straightforward calculations show that

$$\gamma_{N_{j+1}} \geqslant \exp(2\tilde{\beta}\gamma^-\delta)\gamma_{N_j}$$

and

$$\gamma_{l+1} \leqslant \exp(2\tilde{\beta}\delta^+)\gamma_{N_j}.$$

Hence,

$$\mathbb{E}\left\{\left|x_{N_j+1}\right|^2\left|\mathcal{F}_{N_j}\right.\right\} \leqslant \exp(-2\gamma^-\tilde{\beta}\delta^-)\left|x_{N_j}\right|^2 + \exp(2\tilde{\beta}\delta^+)\mathbb{E}\left\{\sum_{l=N_j}^{N_{j+1}-1}\kappa_l\big|\mathcal{F}_{N_j}\right\}$$

and for the required result, we need to bound the last term. By (6.2), we have

$$\mathbb{E}\left(\sum_{l=N_j}^{N_{j+1}-1}\Delta_l\big|\mathcal{F}_{N_j}\right) \leqslant \delta + \Delta_{\max} = \delta^+$$

and we obtain

$$\mathbb{E}\left(\sum_{l=N_j}^{N_{j+1}-1}\kappa_l\big|\mathcal{F}_{N_j}\right) \leqslant [2\tilde{\alpha} + \sigma^2]\delta^+.$$

This gives the desired bound. □

We now proceed to prove the ergodicity and moment bound. We prove geometric ergodicity of the Markov chain $\{y_j, l_j\}$ by using the approach highlighted in Meyn & Tweedie (1992). In particular, we use a slight modification of Theorem 2.5 in Mattingly *et al*. (2002). Inspection of the proof in the appendix of that paper shows that, provided an invariant probability measure exists, and this follows from Lemma 6.4, the set C in the minorization condition need not be compact: it simply needs to be a set to which return times have exponentially decaying tails.

Let

$$P(y, l, A) = \mathbb{P}((y_1, l_1) \in A|(y_0, l_0) = (y, l)),$$

where

$$A \in \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{Z}^+), \quad (y, l) \in \mathbb{R}^m \times \mathbb{Z}^+.$$

We write $A = (A_y, A_l)$ with $A_y \in \mathcal{B}(\mathbb{R}^m)$ and $A_l \in \mathcal{B}(\mathbb{Z}^+)$.

The minorization condition that we use, generalizing that in Lemma 2.5 of Mattingly *et al*. (2002), is now proved.[1]

LEMMA 6.5 Let C be compact. For $\delta > 5\Delta_{\max}$, there is $\zeta > 0, \overline{y} \in \mathbb{R}^m$ and $\varepsilon > 0$ such that

$$P(y, l, A) \geqslant \zeta \nu(A) \quad \forall\, A \in \mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{Z}^+), \quad (y, l) \in C \times \mathbb{Z}^+,$$

where

$$\nu(A) = \mathrm{Leb}\{B(\overline{y}, \varepsilon) \cap A_y\} \cdot \mathbf{1}\{k^\star(\overline{y}) \in A_l\}.$$

Proof. Let $M = N_1(2\Delta_{\max})$ and $N = N_1(\delta)$. Recall the definition (5.1) of $\bar{y}$. Since $t_M \leqslant 3\Delta_{\max}$ almost surely, setting $r^2 = R^2 + B^2$, Corollary 6.3 implies that we can choose positive B and R sufficiently large so that

$$\mathbb{P}\left\{\sup_{0 \leqslant n \leqslant M} |x_n|^2 \leqslant r^2\right\} \geqslant \frac{1}{2}$$

and $\overline{y} \in B(0, r)$ and $C \subseteq B(0, R)$. Label this event, with probability in excess of $\frac{1}{2}$, by E_1. If E_1 occurs, then there exists $l \in \{0, \ldots, M\}$ such that $k_l \leqslant k^*(B(0, r))$. This follows by contradiction, since otherwise $\Delta_j = 2^{j+1}\Delta_{-1}$ for $j \in \{0, \ldots, M\}$ and

$$t_M = \sum_{j=0}^{M-1} \Delta_j \leqslant \Delta_{\max} \sum_{j=0}^{M-1} 2^{j-M} \leqslant \Delta_{\max} \sum_{k=1}^{\infty} 2^{-k} = \Delta_{\max}.$$

However, $t_M \geqslant 2\Delta_{\max}$, a contradiction. Once $k_j \leqslant k^*(B(0, r))$, it follows that $k_n \leqslant k^*(B(0, r))$ for $n \in \{l, \ldots, M\}$ as a consequence of the step-size selection mechanism.

Assume that E_1 has occurred. By choice of ϵ sufficiently small, $B(\overline{y}, \epsilon) \subseteq B(0, r)$. We now choose the η_j for $j \in \{M, \ldots, N-1\}$ to ensure the event E_2, namely, that

$$x_j \in B(\overline{y}, \varepsilon), \quad M + 1 \leqslant j \leqslant N.$$

It is possible to ensure that the event has probability $p_1 > 0$, uniformly for $X \in C$ and $k_0 \in \mathbb{Z}^+$. The fact that $x_M \in B(0, r)$ gives uniformity in $X \in C$. We prove an upper bound on the number of steps after M to get probability independent of $k_{-1} \in \mathbb{Z}^+$. To do this, note that $k_n \leqslant k^*(B(0, r))$ now for $n \in \{j, \ldots, N\}$, again as a consequence of the step-size mechanism. In fact $k_N = k^*(B(\overline{y}, \epsilon)) = k^*(\overline{y})$. This follows because an argument analogous to that above proves that there is $l \in \{M + 1, \ldots, N\}$ for which $k_n \leqslant k^*(B(\overline{y}, \epsilon)) = k^*(\overline{y})$ for $n \in \{l, \ldots, N\}$. Now $k^*(B(\overline{y}, \epsilon)) = k^*(\overline{y})$, by continuity of f and possibly by further reduction of ϵ. Since $k_j < k^*(\overline{y}) = \mathcal{G}(\overline{y}, 1)$ is not possible, it follows that $k_N = k^*(\overline{y})$.

If E_1 and E_2 both occur, then, for some $\gamma > 0$, the probability that $y_1 = x_{N_1(\delta)} \in A_y$ is bounded below by $\gamma\,\mathrm{Leb}\{A_y \cap B(\overline{y}, \varepsilon)\}$, for some $\gamma > 0$, because

$$x_N = x^\star_{N-1} + \sqrt{\Delta_{N-1}}\, g(x^\star_{N-1})\eta_N,$$

[1] Note that although C is compact in the following, $C \times \mathbb{Z}^+$ is not.

x_{N-1} is in a compact set and g is invertible. The fact that η_j are i.i.d Gaussian gives the required lower bound in terms of Lebesgue measure. The final result follows with $\zeta = \gamma p_1/2$. □

With this minorization condition in hand, we turn to the proof of ergodicity.

Proof of Theorem 5.3. The existence of an invariant measure π follows from the Foster–Lyapunov drift condition of Lemma 6.4, which gives tightness of the Krylov–Bogoljubov measures. Lemma 6.4 shows that the chain $\{y_j, l_j\}$ repeatedly returns to the set $C \times \mathbb{Z}^+$ and that the return times have exponentially decaying tails. This generalizes Assumption 2.2 in Mattingly *et al*. (2002). Lemma 6.5 gives a minorization condition enabling a coupling. Together these two results give Theorem 5.3, by applying a straightforward modification of the arguments in Appendix A of Mattingly *et al*. (2002). □

Proof of Theorem 5.4. We define the stopping time N by

$$N = \inf_{n \geqslant 0} \{n : t_n \geqslant T\}$$

noting that

$$T \leqslant t_N \leqslant T + \Delta_{\max}.$$

Note that (2.5) implies that

$$\sup_{0 \leqslant t \leqslant t_N} |\overline{X}(t)|^2 \leqslant C \left[1 + \sup_{0 \leqslant k \leqslant N} |x_k|^2 + \sup_{(t-s) \in [0, \Delta_{\max}], s \in [0,T]} |W(t) - W(s)|^2 \right].$$

Here, we have used the fact that by Lemma 6.1,

$$|x_k^\star|^2 \leqslant |x_k|^2 + 2\Delta_{\max}\tilde{\alpha}.$$

From this relationship between the supremum of moments of $\overline{X}(t)$ and $X(t)$, and from the properties of increments of Brownian motion, it follows that to prove Theorem 5.4, it suffices to bound

$$\mathbb{E} \exp\left(\lambda \sup_{0 \leqslant k \leqslant N} |x_k|^2 \right)$$

for some $\lambda > 0$. However, this follows from the fact that $t_N \leqslant T + \Delta_{\max}$ and Corollary 6.3. □

7. Numerical experiments: pathwise convergence

We now provide some numerical experiments to complement the analysis of the previous sections. We begin, in this section, by demonstrating the importance of Assumption 3.2 in ensuring pathwise convergence. In Section 8, we discuss an abstraction of the method presented and studied in detail in this paper. Section 9 then shows how this abstraction leads to a variant of the method discussed here, tailored to the study of damped-driven Hamiltonian systems. We provide numerical experiments showing the efficiency of the methods at capturing the system's invariant measure.

In the convergence analysis, we made Assumption 3.2, the second part of which was to assume that the hitting time of small neighbourhoods of the set $\Psi(0)$ is large with high probability. We now illustrate

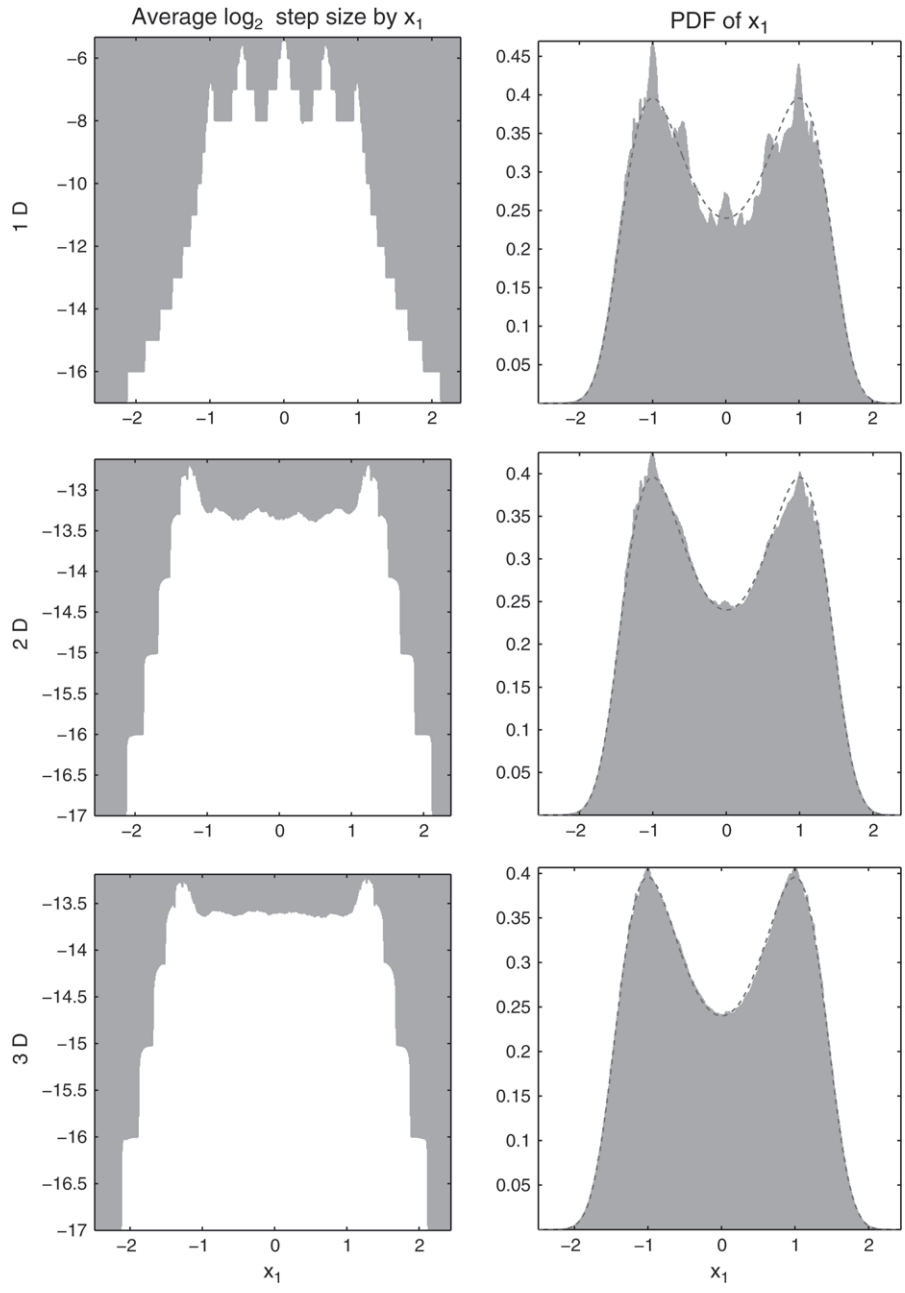

FIG. 1. Effect of the 'bad set' $\Psi(\epsilon)$ in different dimensions. On the right, the numerically obtained density and true analytic density (dashed line). On the left, the average log of the step-size taken versus the spatial position of x_1.

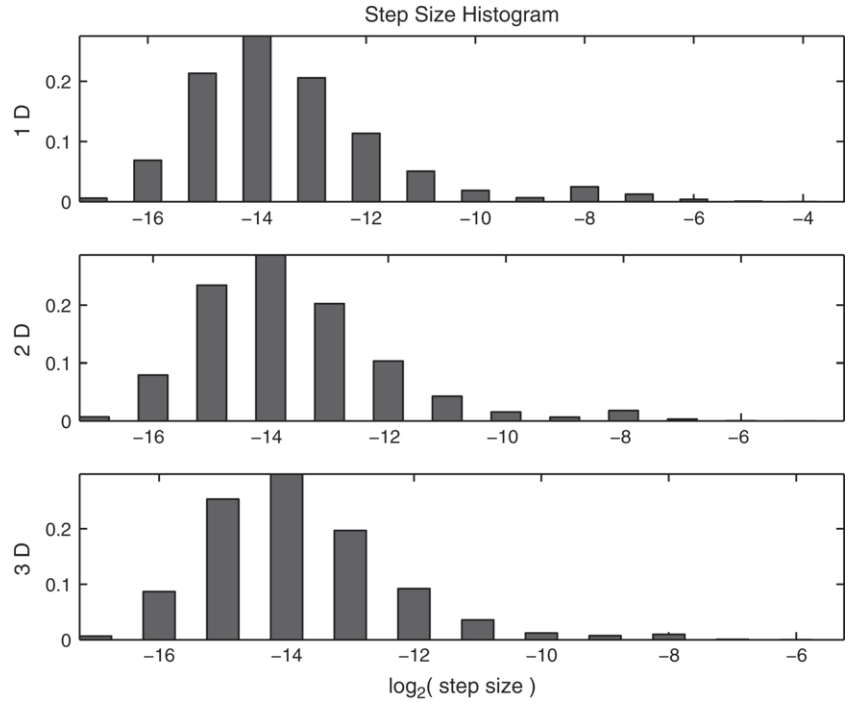

FIG. 2. Gradient problem—Time step histogram.

that this is not simply a technical assumption. We study the test problem

$$\mathrm{d}y = y - y^3 + \mathrm{d}W, \tag{7.1}$$

where W is a real-valued scalar Brownian motion. The set $\Psi(0)$ comprises the points where $f(y) := y - y^3$ satisfies $f(y) = 0$ and $f'(y) = 0$, i.e. the points $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$. Since the problem is 1D, the hitting time to neighbourhoods of these points is not small.

For contrast, we apply the algorithm to the systems in 2D and 3D found by making identical copies of (7.1) in the extra dimensions with each dimension driven by an independent noise. Thus, the set $\Psi(0)$ comprises the tensor product of the set $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$ in the appropriate number of dimensions. Small neighbourhoods of this set do have large hitting time, with high probability. To illustrate the effect of this difference between hitting times, we show, in Fig. 1, the average time step taken at a given location in space for (the first component of) y. Note that in 1D, the algorithm allows quite large average steps in the neighbourhood of the points $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$. This does not happen in dimensions two and three because the probability that the other components of y is also near to the set $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$ at the same time is very small. The effect of this large choice of time steps in 1D is apparent in the empirical densities for (the first component of) y which are also shown in Fig. 1; these are generated by binning 200 paths of the SDE (7.1) over 200 time units. It is important to realize that, although the algorithm in 1D makes a very poor approximation of the empirical density, this occurs only because of a relatively small number of poorly chosen time steps. Figure 2 shows a histogram of the time steps (k_n values) taken in 1D, 2D and

3D. The plots are nearly identical, except that in 1D the algorithm allows the method to take a small number of larger steps with $k_n = 4$.

8. Generalizations of the method

The method given in (2.1) can be seen as a simple instance of a general class of methods based on comparing, with some error metric, one time step given by two methods of the form

$$\begin{aligned} x_{n+1} &= x_n + F(x_n, \Delta_{n+1})\Delta_{n+1} + G(x_n, \Delta_{n+1})\sqrt{\Delta}_{n+1}\eta_n, \\ \overline{x}_{n+1} &= x_n + \overline{F}(x_n, \Delta_{n+1})\Delta_{n+1} + \overline{G}(x_n, \Delta_{n+1})\sqrt{\Delta}_{n+1}\eta_n \ , \end{aligned} \tag{8.1}$$

where $F, \overline{F}, G, \overline{G}$ are deterministic functions. The method in (2.1) was based on comparing the pair of explicit methods given by

$$\begin{aligned} x_{n+1} &= x_n^\star + \sqrt{\Delta}_{n+1} g(x_n)\eta_n, \\ \overline{x}_{n+1} &= \hat{x}_n + \sqrt{\Delta}_{n+1} g(x_n^\star)\eta_n, \end{aligned}$$

where

$$\begin{aligned} x_n^\star &= x_n + \Delta_{n+1} f(x_n), \\ \hat{x}_n &= x_n + \frac{1}{2}\Delta_{n+1}[f(x_n) + f(x_n^\star)]. \end{aligned}$$

In (2.1), closeness was measured by the difference, divided by the step-size, between the conditional expectations of one time step of the two different methods; this gives

$$\frac{2}{\Delta_{n+1}}|\mathbb{E}x_{n+1} - \mathbb{E}\overline{x}_{n+1}| = 2|F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1})| = |f(x_n) - f(x_n^\star)|. \tag{8.2}$$

From this point of view, it is clear that the method discussed thus far is one of the large family of methods. Depending on the setting, one might want to compare methods other than the simple Euler methods used thus far. Also one can consider different error measures. In Section 9, we study a damped-driven Hamiltonian problem and use ideas from symplectic integration to design an appropriate method. In the discussion at the end of the article, we return to the question of different error measures.

9. Numerical experiments: long-time simulations

In this section, we demonstrate that the ideas established for the rather specific adaptive scheme studied, and for the particular hypotheses on the drift and diffusion, extend to a wider class of SDEs and adaptive methods.

As a test problem, we consider the Langevin equation

$$\begin{aligned} \mathrm{d}q &= p\,\mathrm{d}t, \\ \mathrm{d}p &= -[\delta(q)p + \Phi'(q)]\mathrm{d}t + g(q)\mathrm{d}W, \end{aligned} \tag{9.1}$$

where $2\delta(q) = g^2(q)$,

$$\Phi(q) = \frac{1}{4}(1 - q^2)^2 \quad \text{and} \quad g(q) = \frac{4(5q^2 + 1)}{5(q^2 + 1)}.$$

The preceding theory does not apply to this system, since it is not uniformly elliptic; furthermore, it fails to satisfy (1.2). However, it does satisfy a Foster–Lyapunov drift condition and since it is hypoelliptic the equation itself can be proven geometrically ergodic (Mattingly *et al*., 2002). In Mattingly *et al*. (2002), it was shown that the implicit Euler scheme was ergodic when applied to (9.1), and a similar analysis would apply to a variety of implicit methods. Since the adaptive schemes we study in this section enforce closeness to such implicit methods, we believe that analysis similar to that in the Section 8 will extend to this Langevin equation and to the adaptive numerical methods studied here.

We will compare two different methods based on different choices of the stepping method. The first is the Euler-based scheme given in (2.1). The second is the following scheme:

$$\begin{aligned}
q_{n+1} &= q_n^\star, \\
p_{n+1} &= p_n^\star + g(q_n^\star)\sqrt{\Delta_n}\eta_{n+1}, \\
\overline{q}_{n+1} &= q_n + \left(\frac{p_n + p_n^\star}{2}\right)\Delta_n, \\
\overline{p}_{n+1} &= p_n - \Phi'\left(\frac{q_n + q_n^\star}{2}\right)\Delta_n - \delta\left(\frac{q_n + q_n^\star}{2}\right)\left(\frac{p_n + p_n^\star}{2}\right)\Delta_n \\
&\quad + g\left(\frac{q_n + q_n^\star}{2}\right)\sqrt{\Delta_n}\eta_{n+1},
\end{aligned} \tag{9.2}$$

where

$$\begin{aligned}
q_n^\star &= q_n + p_n\Delta_n, \\
p_n^\star &= p_n - \Phi'(q_n^\star)\Delta_n - \delta(q_n^\star)p_n\Delta_n.
\end{aligned}$$

Once again we will use the comparisons between the two updates (with and without bars) to control the error. As before, we control on the difference in the expected step. The point of the particular form used here is that, in the absence of noise and damping, the adaptation constrains the scheme to take steps which are close to those of the symplectic midpoint scheme, known to be advantageous for Hamiltonian problems; if the noise and damping are small, we expect the Hamiltonian structure to be important.

In both the Euler and the Symplectic case, the stepping methods take the form

$$\begin{aligned}
q_{n+1} &= q_n + f_n^{(1)}\Delta_n, \\
p_{n+1} &= p_n + f_n^{(2)}\Delta_n + g_n\sqrt{\Delta_n}\eta_{n+1}, \\
\overline{q}_{n+1} &= \overline{q}_n + \overline{f}_n^{(1)}\Delta_n, \\
\overline{p}_{n+1} &= \overline{p}_n + \overline{f}_n^{(2)}\Delta_n + \overline{g}_n\sqrt{\Delta_n}\eta_{n+1},
\end{aligned} \tag{9.3}$$

where f_n, $\overline{f}_n$, g_n and $\overline{g}_n$ are adapted to $\mathcal{F}_{n-1}$. In this notation, the metric becomes

$$[(\overline{f}_n^{(1)} - f_n^{(1)})^2 + (\overline{f}_n^{(2)} - f_n^{(2)})^2]^{\frac{1}{2}} < \tau.$$

In the remainder of this section, we present numerical experiments with the two methods just outlined. We study the qualitative approximation of the invariant measure, quantify this approximation and measure its efficiency and study the behaviour of time steps generated.

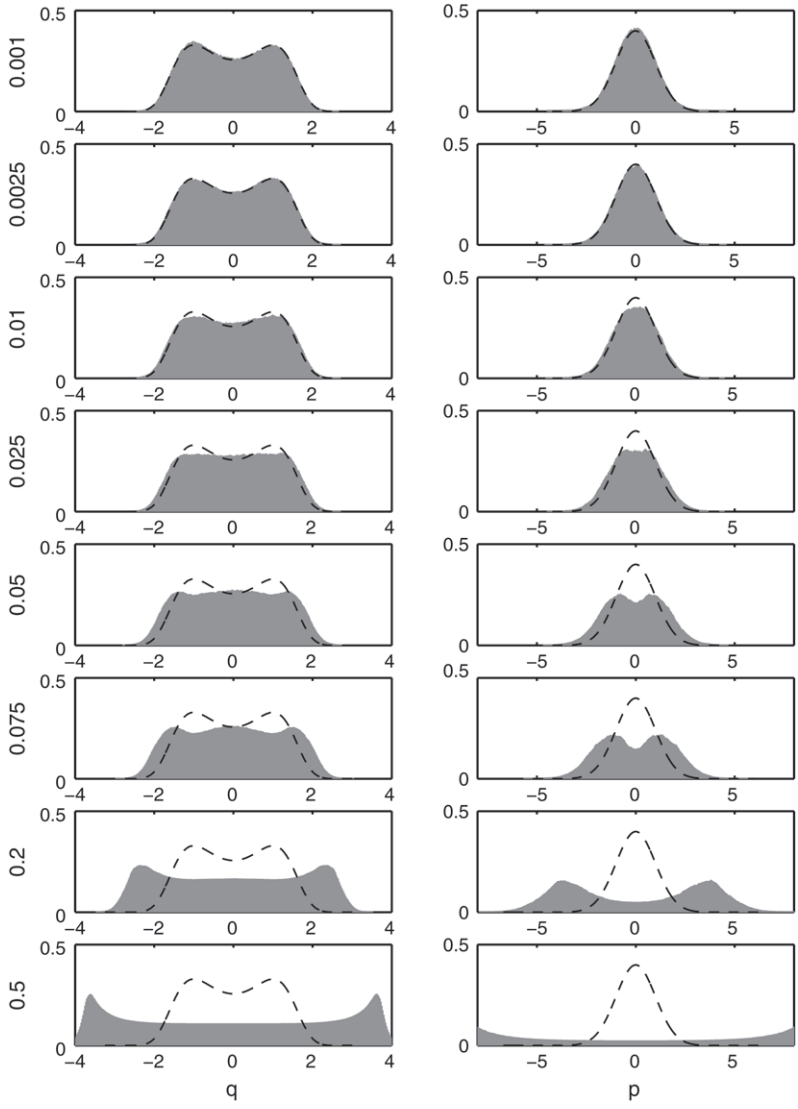

FIG. 3. Distribution of q, p with different τ for Euler method. The value of tolerance τ is on the left of each figure.

Figure 3 plots the numerically obtained time average of the position p and momentum q for various values of the tolerance τ. The dotted lines are the true invariant measure of the underling SDE which can be computed analytically in this particular case. Note that the method appears stable for all values of τ, in contrast to the forward Euler method which blows up when applied to this equation. Though apparently stable, the results are far from the true distribution for large τ. Figure 3 gives the analogous plots for the adaptive symplectic method given in (9.3). Note that these methods seem to do a much better job of reproducing the invariant measure faithfully at large τ.

It is also important to study accuracy per unit of computational effort. Figure 5 gives plots of the error in the total variation norm (the L^1 distance between the numerically computed time averages

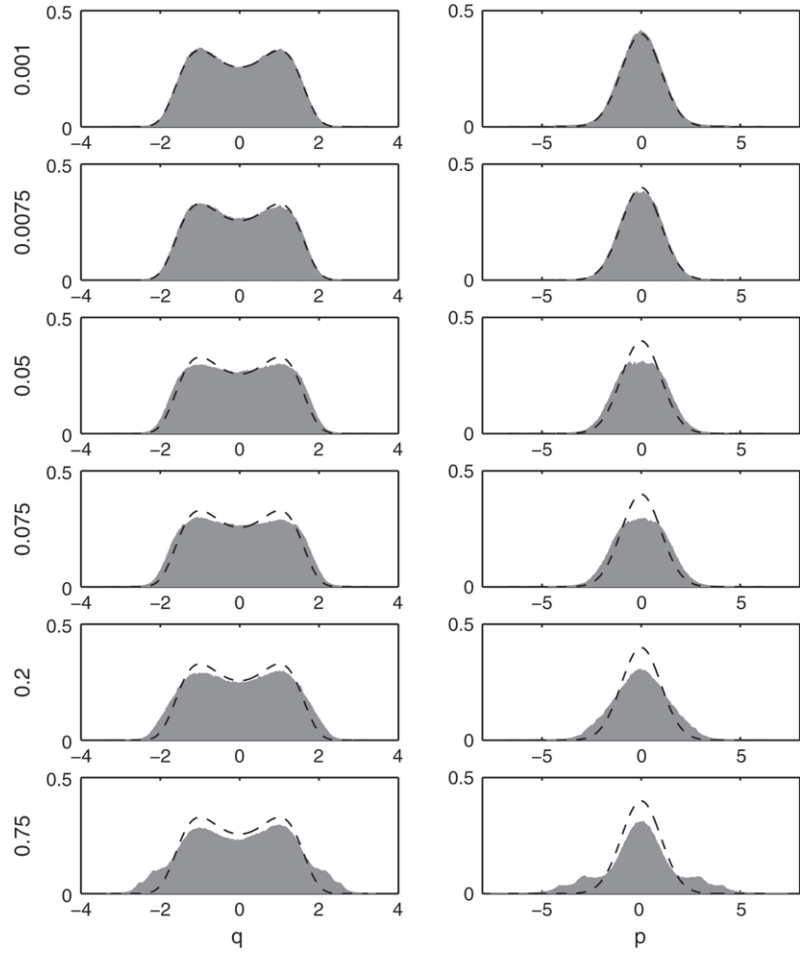

FIG. 4. Distribution of q, p with different τ for the symplectic method. The value of tolerance τ is on the left of each figure.

and the exact analytic answer) versus the τ used and versus the steps per unit of time; the latter provides a measure of unit cost. The top plots are for the momentum q and the bottom for the position p. The plots on the right also include two fixed-step methods, one using the simple forward Euler scheme and the second using the first of the symplectic schemes. The fixed-step Euler scheme blows up for steps larger than those given. We make the following observations on the basis of this experiment:

- The fixed-step symplectic method is the most efficient at small time steps.
- The adaptive symplectic method is considerably more efficient than the adaptive and fixed-step Euler methods.
- The adaptive symplectic method is the most robust method, providing reasonable approximations to the invariant density for a wide range of τ.

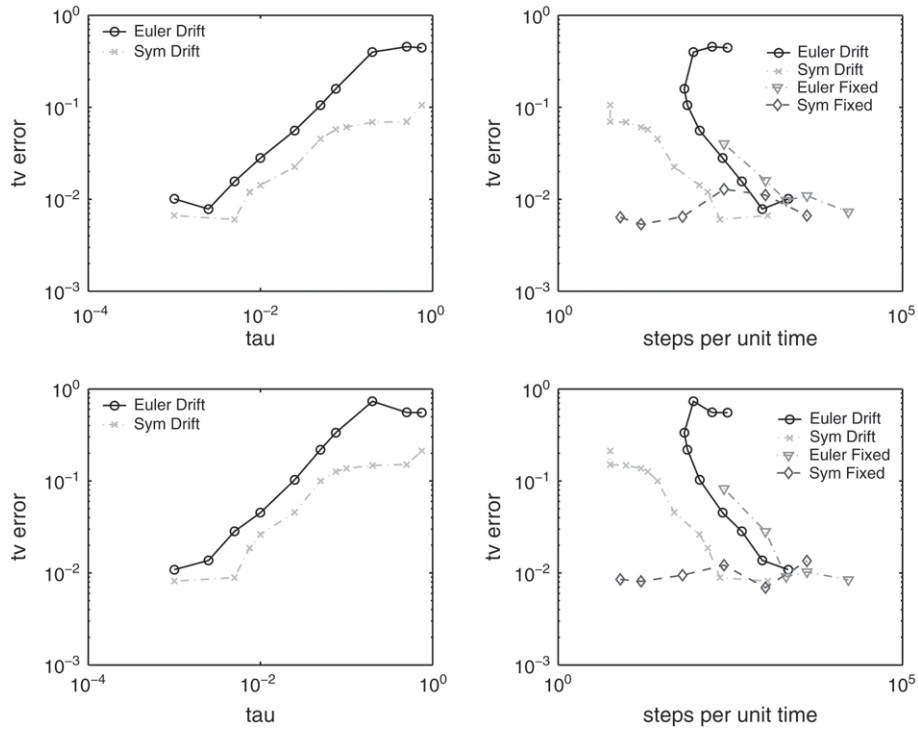

FIG. 5. Total variation error versus τ for position (top left) and the momentum (bottom left). Total variation error versus τ for steps per unit time (top right) and the momentum (bottom right).

Note that the adaptive methods have not been optimized and with careful attention might well beat the fixed-step methods, both as measured by accuracy per unit cost, as well as by robustness. Further study of this issue is required.

10. Conclusions

This paper proposes a simple adaptive strategy for SDEs which is designed to enforce geometric ergodicity, when it is present in the equation to be approximated; without adaptation, methods such as explicit Euler may destroy ergodicity. As well as proving ergodicity, we also prove some exponential moment bounds on the numerical solution, again mimicking those for the SDE itself. Furthermore, we prove finite-time convergence of the numerical method; this is nontrivial because we do not assume (and it is not true in general) that the time-step sequence tends to zero with user-input tolerance. It would be of interest to transfer this finite-time convergence to a result concerning convergence of the invariant measures, something which is known for fixed time-step schemes (Talay, 1990, 1999, 2002).

As discussed in Section 8, the scheme we study in detail here is prototypical of more advanced schemes comparing two more sophisticated methods and controlling both on drift and on diffusion. Here, we have mainly used simple forward Euler methods and controlled only on the drift: our error measure is based on the conditional means. The split-step approach we take allows for additional terms to be added to the error measure, to ensure that the diffusion step is also controlled. The general idea

is to enforce the closeness of one step by two different methods. One has freedom in the choice of the methods and the measure of closeness. We now briefly mention to other error measures which make this idea specific.

For simplicity, let us assume work in 1D though the ideas generalize directly to higher dimensions. The simple error control given in (8.2) controls only the difference in the expectation of one step of the two methods. However, one can also use measure which ensures the closeness of the entire distribution of one time step of the two methods. Given $x_n = \bar{x}_n$ and Δ_{n+1}, one step of a method of the form (8.1) is Gaussian. Hence, it is reasonable to require that the standard deviations are close to each other. The error criterion would then be

$$\frac{1}{\Delta_{n+1}}|\mathbb{E}x_{n+1} - \mathbb{E}\bar{x}_{n+1}| + \frac{1}{\sqrt{\Delta_{n+1}}}|\mathrm{SD}(x_{n+1}) - \mathrm{SD}(\bar{x}_{n+1})|$$
$$= |F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1})| + ||G(x_n, \Delta_{n+1})| - |\bar{G}(x_n, \Delta_{n+1})|| < \tau.$$

In some ways, comparing the mean and standard deviations is rather arbitrary. A more rational choice might be to ensure the closeness of the total variation distance of the densities after one time step of the two methods. A simple way to do this is to compare the relative entropy of the two distributions. Since the distributions are Gaussian, this can be done explicitly. One finds that the criterion based on controlling relative entropy per unit step is

$$\frac{(F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1}))^2}{G(x_n, \Delta_{n+1})^2} + \left(\frac{\bar{G}(x_n, \Delta_{n+1})^2}{G(x_n, \Delta_{n+1})^2} - 1\right) - \log\frac{\bar{G}(x_n, \Delta_{n+1})^2}{G(x_n, \Delta_{n+1})^2} < 2\tau\ .$$

It is interesting to note that this measure correctly captures the fact that one should measure the error in the drift on the scale of the variance. In other words, if the variance is large, one does not need to be as accurate in calculating the drift as it will be washed out by the noise anyway. Since the above measure is expensive to calculate, one can use the fact that $\frac{\bar{G}}{G} - 1$ is small to obtain the asymptotically equivalent criterion

$$\frac{(F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1}))^2}{G(x_n, \Delta_{n+1})^2} + \frac{1}{2}\left(\frac{\bar{G}(x_n, \Delta_{n+1})^2}{G(x_n, \Delta_{n+1})^2} - 1\right)^2 < 2\tau.$$

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Appendix A. Two exponential martingale estimates in discrete time

Let $\{\mathcal{F}_n, n \geqslant 0\}$ be a filtration. Let η_k be a sequence of random variables with η_k adapted to $\mathcal{F}_k$ and such that η_{k+1} conditioned on $\mathcal{F}_k$ is normal with mean zero and variance $\sigma_k^2 = \mathbb{E}[\eta_{k+1}^2|\mathcal{F}_k] < \infty$. We define the following processes:

$$M_n = \sum_{k=1}^{n} \eta_k, \quad \tilde{M}_n = \sum_{k=1}^{n} \eta_k^2 - \sigma_{k-1}^2,$$
$$\langle M\rangle_n = \sum_{k=0}^{n-1} \sigma_k^2, \quad \langle \tilde{M}\rangle_n = \sum_{k=0}^{n-1} 2\sigma_k^4.$$

As the notation suggests, $\langle M\rangle_n$ and $\langle \tilde{M}\rangle_n$ are the quadratic variation processes in that $M_n^2 - \langle M\rangle_n$ and $\tilde{M}_n^2 - \langle \tilde{M}\rangle_n$ are local martingales with respect to $\mathcal{F}_n$.

LEMMA A.1 Let $\alpha > 0$ and $\beta > 0$, then the following estimate holds:

$$\mathbb{P}\left(\sup_k \left(M_k - \frac{\alpha}{2}\langle M\rangle_k\right) \geqslant \beta\right) \leqslant \mathrm{e}^{-\alpha\beta}.$$

If in addition $\sigma_k^2 \leqslant \sigma_*^2 \in \mathbb{R}$ for all $k \in \mathbb{N}$ almost surely, then

$$\mathbb{P}\left(\sup_k \left(\tilde{M}_k - \frac{\alpha}{2}\langle \tilde{M}\rangle_k\right) \geqslant \beta\right) \leqslant \mathrm{e}^{-\frac{\beta}{\lambda^2}},$$

where $\lambda^2 = 2\sigma_*^2 + 1/\alpha$.

Proof of Lemma A.1. We begin with the first estimate. Define $N_n = \exp(\alpha M_n - \frac{\alpha^2}{2}\langle M\rangle_n)$ and observe that $N_n = \mathbb{E}\{N_{n+1}|\mathcal{F}_n\}$. This in turn implies that $\mathbb{E}|N_n| = \mathbb{E}N_n = N_0 = 1 < \infty$. Combining these facts, we see that N_n is a martingale. Hence, the Doob–Kolmogorov martingale inequality (Rogers & Williams, 2000) implies

$$\mathbb{P}\left(\sup_n N_n > c\right) \leqslant \frac{\mathbb{E}N_0}{c} = \frac{1}{c}.$$

Since $\mathbb{P}(\sup_n(M_n - \frac{\alpha}{2}\langle M\rangle_n) \geqslant \beta) = \mathbb{P}(\sup_n N_n > \mathrm{e}^{\alpha\beta})$, the proof is complete.

The second estimate is obtained in the same way after some preliminary calculations. We define $\phi(x) = \frac{1}{2}\ln(1-2x)$ and $\psi(x,b) = -x - bx^2$. Observe that $c\psi(x,b) = \psi(cx, b/c)$ and $\phi(x) \geqslant \psi(x,b)$, if $x \in [0, \frac{1}{2}(\frac{b-1}{b})]$ and $b > 1$. Now

$$\mathbb{P}\left(\sup_n \left(\tilde{M}_n - \frac{\alpha}{2}\langle \tilde{M}\rangle_n\right) \geqslant \beta\right) = \mathbb{P}\left(\sup_n \sum_{k=1}^{n} \frac{\eta_k^2}{\lambda^2} + \frac{1}{\lambda^2}\psi(\sigma_{k-1}^2, \alpha) \geqslant \frac{\beta}{\lambda^2}\right).$$

Setting $\lambda^2 = 2\sigma_*^2 + \frac{1}{\alpha}$, we have that $\frac{1}{\lambda^2}\psi(\sigma_k^2, \alpha) = \psi(\frac{\sigma_k^2}{\lambda^2}, \lambda^2\alpha) \leqslant \phi(\frac{\sigma_k^2}{\lambda^2})$ for all $k \geqslant 0$ since $\sigma_k^2 \leqslant \sigma_*^2$ and $\lambda^2\alpha > 1$. Defining

$$\tilde{N}_n = \exp\left(\sum_{k=1}^{n} \frac{\eta_k^2}{\lambda^2} + \phi\left(\frac{\sigma_{k-1}^2}{\lambda^2}\right)\right),$$

we have

$$\mathbb{P}\left(\sup_n \left(\tilde{M}_n - \frac{\alpha}{2}\langle \tilde{M}\rangle_n\right) \geqslant \beta\right) \leqslant \mathbb{P}\left(\sup_n \tilde{N}_n \geqslant \mathrm{e}^{\frac{\beta}{\lambda^2}}\right).$$

Now recall that if ξ is a unit Gaussian random variable, then $\mathbb{E}\exp(c\xi^2) = 1/\sqrt{1-2c}$ for $c \in (-\frac{1}{2}, \frac{1}{2})$. By construction, $\frac{\eta_k}{\lambda}$, conditioned on $\mathcal{F}_{k-1}$, is a Gaussian random variable with variance less than $\frac{1}{2}$. Hence,

$$\mathbb{E}\left(\exp\left(\frac{\eta_k^2}{\lambda^2}\right)\middle|\mathcal{F}_{k-1}\right) = \exp\left(-\phi\left(\frac{\sigma_{k-1}^2}{\lambda^2}\right)\right).$$

Using this, one sees that $\mathbb{E}\{\tilde{N}_{n+1}|\mathcal{F}_n\} = \tilde{N}_n$ and $\mathbb{E}|\tilde{N}_n| = 1 < \infty$, hence $\tilde{N}_n$ is a martingale. By the same argument as before using the Doob–Kolmogorov martingale inequality, we obtain the quoted result. □