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SINGULAR LIMITS IN FREE BOUNDARY PROBLEMS

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ABSTRACT. We analyze the following class of nonlinear eigenvalue problems: find $(\underline{u}, \mu) \in B \times \Re$ satisfying

$$D\underline{u} + \mu H(\underline{a} \cdot \underline{u} - 1)\underline{f}(\underline{u}) = 0 \quad \text{in } \Omega \subseteq \Re^N, \tag{1}$$

$$\underline{u} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Here $H(X)$ is the Heaviside step-function defined by

$$H(X) = 0, \qquad X \leq 0$$
$$H(X) = 1, \qquad X > 0.$$

B is some Banach space appropriate to the problem. D is taken to be a (possibly nonlinear) differential operator with the property that, when $\mu = 0$, equations (1,2) have the unique solution $\underline{u} \equiv 0$.

The problem (1,2) is of free boundary type, since determination of the sets on which $\underline{a} \cdot \underline{u} = 1$ is necessary to determine the solution. Our motivation is the study of porous medium combustion where equations of the form (1,2) represent equilibrium states of the coupled chemical and heat-transfer processes governing the combustion [**3, 4**]. The step function H arises from the diffusion limited reaction rate, in the limit of large activation energy. The reaction behaves as a switch, triggered when the temperature of the solid phase reaches a threshold value. Problems such as (1,2) also arise in a variety of other applications such as the study of vortex motion in ideal fluids [**1**] and plasma physics [**7**].

Clearly $\underline{u} \equiv 0$ satisfies (1,2) for all μ. However, there is no classical bifurcation from this trivial solution, since all other solutions must satisfy $\underline{a} \cdot \underline{u} > 1$ at some point in Ω, and hence cannot be of arbitrarily small supremum norm. Nonetheless, it is useful to develop a constructive approach to the solution of (1,2) since explicit constructions are useful both as the basis for numerical continuation procedures and as the basis for local time-dependent stability calculations.

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Let $\chi = \{x \in \Omega | \underline{a} \cdot \underline{u} > 1\}$. Then we find that, in a large class of problems, the singular limit meas$(\chi) \to 0$ provides a basis for *constructive* existence proofs for nontrivial solutions of (1,2). The relevance of this limit in nonconstructive existence theory, when D is a linear elliptic operator, was noticed in the case $N = 1$ in [**2**] and $N = 2$ in [**1**].

In [**3,6**] a rigorous connection is made between the limiting behavior of (1,2) as meas$(\chi) \to 0$ and the application of standard local bifurcation theory to a transformed problem. The transformed problem arises from a coordinate transformation which maps χ onto the unit ball; considering meas$(\chi) = 0$ creates an artificial trivial solution and local bifurcation theory can be used to construct small amplitude solutions of the transformed problem, for meas$(\chi) \ll 1$.

This connection allows the results of local bifurcation theory to be used to construct solutions of (1,2), even though there is no local bifurcation structure inherent in (1,2). In [**3**] the approach is applied to proving the existence of traveling waves in a fourth order partial differential equation modeling full porous medium combustion; in [**6**] the approach is applied to the existence of steady solutions of a simpler model elliptic problem, derived from the equations in [**3**] under various simplifying assumptions (which are detailed in [**5**]).

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