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Volterra integral equations and a new Gronwall inequality (Part I: The linear case)

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Synopsis

We present a generalisation of the continuous Gronwall inequality and show its use in bounding solutions of discrete inequalities of a form that arise when analysing the convergence of product integration methods for Volterra integral equations. We then use these ideas to prove convergence of a numerical method which is effective in approximating Volterra integral equations of the second kind with weakly singular kernels.

1. Introduction

This paper is concerned with the application of numerical methods to find approximations to the solutions of Volterra integral equations of the form

$$y(t)=g(t)+\int_0^t H(t,s,y(s))\,ds. \tag{1.1}$$

In particular, we are interested in the use of variable mesh methods which arise when transformations of the dependent variable are made to eliminate singularities inherent in the solution of certain equations of the form (1.1).

Consider, for example, a mesh which has a step size of $O(h)$ throughout most of the interval $[0, t]$, but which involves points with the larger step size of $O(h^{\frac{1}{2}})$ over a (smaller) part of the interval. A naive analysis of the consistency and convergence of schemes on such graded meshes, based on the assumption that the mesh spacing is uniformly bounded above by $O(h^{\frac{1}{2}})$, leads to weak results which do not reflect the true order of consistency of the scheme and may not even yield a convergence result at all. Sharper results can often be obtained by the use of more sophisticated analysis of the discrete inequalities which determine the error in the numerical scheme at each mesh point (see [**8**], [**5**]).

We describe a new numerical method for weakly singular Volterra integral equations which is based on the introduction of a variable mesh. The method is easy to apply and the analysis of its convergence is made possible by use of a new Gronwall inequality.

In Section 2 we present a generalisation of the continuous Gronwall inequality of Bellman [2] and show how this may be used to derive general criteria for bounding the errors arising in a numerical scheme for equation (1.1).

In Section 3 we consider singular linear Volterra equations of the form

$$y(t) = g(t) + \int_0^t \frac{y(s)}{(t-s)^{\frac{1}{2}}}\, ds. \tag{1.2}$$

Developing an idea introduced by Norbury in [10], we present a numerical method which involves the transformation of the independent variables, followed by an application of the product trapezium rule. Then we use the criteria of Section 2 to prove convergence of the scheme. We find it to be of $O(h^{\frac{3}{2}-\varepsilon})$, for arbitrary $\varepsilon > 0$, a significant improvement on the direct application of the trapezium rule to equation (1.3), which results in $O(h^{\frac{1}{2}})$ convergence [4]. In addition, the method compares favourably with the $O(h^2)$ non-polynomial spline collocation methods proposed by Brunner [3] in that, although its order of convergence is less, our method is easier to implement in practice.

In analysing only linear equations, we have adopted the view taken in Brunner [4] that any convergence proof for non-linear equations will always involve a linearisation step, and subsequently the structure of the analysis will remain unchanged. In Part II of this paper [11] we will consider a particular class of singular non-linear Volterra equations for which the transformation methods discussed in Section 3 are applicable. The non-linear behaviour, although not critical to the convergence results, is crucial in determining the range of existence and the global stability of the solution.

2. A continuous Gronwall inequality and discrete error equations

The following theorem provides a generalisation of the Gronwall inequality of Bellman [2].

Theorem 2.1 (A new Gronwall inequality). *If the functions $u(y)$ and $b(.,y)$ are non-negative and integrable over $y \in [0, T]$ and if in addition $a \geqq 0$, $\frac{\partial b}{\partial y}(.,y) \geqq 0$ and $b(y,y)$ exists, then*

$$u(t) \leqq a + \int_0^t b(\xi, t)u(\xi)\, d\xi \tag{2.1}$$

implies that

$$u(t) \leqq a \exp\left[\int_0^t b(\xi, t)\, d\xi\right].$$

Proof. Let

$$X(t) = \int_0^t b(\xi, t)u(\xi)\, d\xi. \tag{2.2}$$

Then

$$X'(t) = b(t,t)u(t) + \int_0^t b_y(\xi, t)u(\xi)\, d\xi$$

and so, by equations (2.1) and (2.2)

$$X'(s) \leqq b(s, s)[a + X(s)] + \int_0^s b_y(\xi, s)u(\xi)\, d\xi. \tag{2.3}$$

Now, defining

$$I = \frac{d}{ds}\left[\exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\} X(s)\right]$$

we have

$$I = \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[X'(s) - \left(b(s, s) + \int_0^s b_y(\xi, s)\, d\xi\right) X(s)\right].$$

Applying inequality (2.3) gives us

$$\begin{aligned} I \leqq{} & \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[b(s, s)a + \int_0^s b_y(\xi, s)u(\xi)\, d\xi\right] \\ & - \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[\int_0^s b_y(\xi, s)\, d\xi\right] X(s). \end{aligned}$$

Replacing $X(s)$ by $X(s) + a - a$ in the last term, we observe that

$$\begin{aligned} I \leqq{} & \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[ab(s, s) + a\int_0^s b_y(\xi, s)\, d\xi\right] \\ & - \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[\int_0^s b_y(\xi, s)[X(s) + a - u(\xi)]\, d\xi\right]. \end{aligned} \tag{2.4}$$

But, since $0 \leqq \xi \leqq s$ and $\dfrac{\partial b}{\partial y}(.\,, y) \geqq 0$, we have

$$\begin{aligned} u(\xi) &\leqq a + \int_0^\xi b(\eta, \xi)u(\eta)\, d\eta \\ &\leqq a + \int_0^\xi b(\eta, s)u(\eta)\, d\eta \\ &\leqq a + \int_0^s b(\eta, s)u(\eta)\, d\eta \\ &= a + X(s). \end{aligned}$$

Hence

$$\int_0^s b_y(\xi, s)[X(s) + a - u(\xi)]\, d\xi \geqq 0.$$

Thus, by inequality (2.4)

$$I \leqq \exp\left\{-\int_0^s b(\xi, s)\, d\xi\right\}\left[ab(s, s) + a\int_0^s b_y(\xi, s)\, d\xi\right].$$

Integrating this expression, with respect to s, between 0 and t gives

$$\exp\left\{-\int_0^t b(\xi, t)\, d\xi\right\}X(t) - X(0) \leqq a\left[1 - \exp\left\{-\int_0^t b(\xi, t)\, d\xi\right\}\right].$$

Finally, since $X(0) = 0$, we obtain

$$X(t) \leqq a\left[\exp\left\{\int_0^t b(\xi, t)\, d\xi\right\} - 1\right]$$

and so, by inequality (2.1), we have

$$u(t) \leqq a \exp\left\{\int_0^t b(\xi, t)\, d\xi\right\},$$

which is the required result. □

We now use this theorem to derive bounds on the errors arising in numerical methods for Volterra integral equations. Consider the true solution of an integral equation of the form (1.1), $y(t_i)$, and the numerical approximation to this solution, y_i. We define the approximation error, ε_i by $\varepsilon_i = (y_i - y(t_i))$. The analysis of product integration type methods frequently leads to an inequality for the approximation errors ε_i of the form

$$\varepsilon_i \leqq \delta + h \sum_{j=1}^{i} d_{ij}\varepsilon_j. \tag{2.5}$$

Here h is some measure of the magnitude of the step-size and δ is the consistency error which will typically be in the form of h raised to some power. In the case where uniform mesh-spacing is used on non-singular equations, the d_{ij}'s will be uniformly bounded above independently of h and application of the standard discrete Gronwall inequality will yield the result that

$$|\varepsilon_i| \leqq C\delta, \tag{2.6}$$

where C is independent of h. However, when graded meshes are used or when product integration techniques are applied directly to equations with weakly singular kernels, the case arises where the discrete kernel d_{ij} is not uniformly bounded above independently of h. In these cases more sophisticated analysis is required to yield results of the form (2.6). Using the ideas of iterated kernels from integral equation theory, Dixon and McKee **[5]** have derived a set of conditions on the discrete kernel d_{ij} which enable a result of the form (2.6) to be deduced from inequality (2.5). Essentially the idea behind the derivation of their conditions is to iterate the discrete kernels until they are uniformly bounded above independently of h and then to apply the standard discrete Gronwall lemma. We now present a set of conditions on the discrete kernel which do not require iteration until uniform boundedness is obtained with respect to h. We show in Section 3 how these conditions may be used in practice when proving convergence of a particular numerical scheme.

We define the μ^{th} iterated kernel of d_{ij}, denoted $d_{ij}^{(\mu)}$, by

$$d_{ij}^{(1)} = \frac{|d_{ij}|}{|1 - hd_{ii}|} \tag{2.7}$$

and

$$d_{ij}^{(\mu)} = h \sum_{\ell=j+1}^{i-1} d_{i\ell}^{(1)} d_{\ell j}^{(\mu-1)} \quad \text{for} \quad \mu \geqq 2. \tag{2.8}$$

We employ the function $f^{(\mu)}$, satisfying

$$d_{ij}^{(\mu)} \leqq f^{(\mu)}(h(j+1), hi), \tag{2.9}$$

and define q by

$$q = h \sum_{j=1}^{i-1} d_{ij}^{(1)}. \tag{2.10}$$

THEOREM 2.2. *Let the kernel d_{ij} in the inequality (2.5), its iterates defined by (2.7, 8) and the function $f^{(\mu)}$ defined by inequality (2.9) satisfy the following conditions:*

(i) $hd_{ii} \neq 1$,

(ii) $h \sum_{j=1}^{i-1} d_{ij}^{(1)}$ *is bounded independently of* h,

(iii) *there exists an integer* μ, *independent of* h, *such that* $f^{(\mu)}(t, t)$ *exists,* $\frac{\partial f^{(\mu)}}{\partial s}(s, t) \leqq 0$ *and* $\frac{\partial f^{(\mu)}}{\partial t}(s, t) \geqq 0$,

(iv) $\int_0^t f^{(\mu)}(s, t)\, ds$ *exists and is independent of h. Then inequality* (2.5) *implies inequality* (2.6) *where C is independent of h.*

Note. The actual determination of the function f defined by inequality (2.9) will be made clearer in Section 3, where an example is considered (Lemma 3.2).

Proof. By collecting terms in ε_i and taking moduli, inequality (2.5) gives us

$$|\varepsilon_i| \leqq C_1\delta + h \sum_{j=1}^{i-1} d_{ij}^{(1)} |\varepsilon_j|, \tag{2.11}$$

where C_1 is independent of h. We now show by induction that ε_i satisfies

$$|\varepsilon_i| \leqq C_1 \left(\frac{q^\mu - 1}{q - 1}\right) \delta + h \sum_{j=1}^{i-1} d_{ij}^{(\mu)} |\varepsilon_j|. \tag{2.12}$$

Clearly inequality (2.12) is satisfied for $\mu = 1$, for then it reduces to inequality (2.11). We now derive the inductive step. Assuming that inequality (2.12) holds, we multiply through by $hd_{\ell i}^{(1)}$ and sum over i. This yields

$$h \sum_{i=1}^{\ell-1} d_{\ell i}^{(1)} |\varepsilon_i| \leqq C_1\delta \left(\frac{q^\mu - 1}{q - 1}\right) q + h \sum_{i=1}^{\ell-1} hd_{\ell i}^{(1)} \sum_{j=1}^{i-1} d_{ij}^{(\mu)} |\varepsilon_j|.$$

Using inequality (2.11), we obtain

$$|\varepsilon_\ell| \leqq C_1\delta \left(\frac{q^{\mu+1} - 1}{q - 1}\right) + h \sum_{i=1}^{\ell-1} hd_{\ell i}^{(1)} \sum_{j=1}^{i-1} d_{ij}^{(\mu)} |\varepsilon_j|.$$

Reversing the order of summation yields

$$|\varepsilon_\ell| \leqq C_1\delta\left(\frac{q^{\mu+1}-1}{q-1}\right) + h\sum_{j=1}^{l-2}\left\{\sum_{i=j+1}^{l-1} hd_{\ell i}^{(1)}d_{ij}^{(\mu)}\right\}|\varepsilon_j|.$$

Interchanging the roles of the indexes ℓ and i, and noting that $d_{ij-1}^{(\mu)}$ is positive, we get

$$|\varepsilon_i| \leqq C_1\delta\left(\frac{q^{\mu+1}-1}{q-1}\right) + h\sum_{j=1}^{i-1}\left\{\sum_{\ell=j+1}^{i-1} hd_{i\ell}^{(1)}d_{\ell j}^{(\mu)}\right\}|\varepsilon_j|.$$

Using the definition of $d_{ij}^{(\mu)}$ given in (2.8), we see that the induction is complete. By using definition (2.9) in inequality (2.12), we obtain

$$|\varepsilon_i| \leqq C_1\delta\left(\frac{q^{\mu}-1}{q-1}\right) + h\sum_{j=1}^{i-1} f^{(\mu)}(h(j+1), hi)\,|\varepsilon_j|. \tag{2.13}$$

We now modify (2.13) so that we may use the continuous Gronwall inequality of Theorem 2.1. The derivation of discrete inequalities from their corresponding continuous results is discussed in **[6]**. Following this, we set $x(s) = |\varepsilon_j|$ for $s \in [t_j, t_{j+1})$, with $t_j = jh$. Noting that $f_s(s, t) \leqq 0$, (2.13) gives us

$$x(t_i) \leqq C_1\left(\frac{q^{\mu}-1}{q-1}\right)\delta + \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}} f^{(\mu)}(s, t_i)x(s)\,ds,$$

so that

$$x(t_i) \leqq C_1\left(\frac{q^{\mu}-1}{q-1}\right)\delta + \int_0^{t_i} f^{(\mu)}(s, t_i)x(s)\,ds.$$

By applying the result of Theorem 2.1, we obtain

$$|\varepsilon_i| = x(t_i) \leqq C_1\left(\frac{q^{\mu}-1}{q-1}\right)\exp\left\{\int_0^{t_i} f^{(\mu)}(s, t_i)\,ds\right\}\delta.$$

Since q, μ and $\int_0^{t_i} f^{(\mu)}(s, t_i)\,ds$ are all independent of h, we have the result that

$$\varepsilon_i \leqq C\delta,$$

where C is independent of h.

3. Abel-type singularities

In this section we present and prove convergence of a numerical method for singular equations of the form (1.2). To avoid the poor convergence results which are obtained when applying product integration methods on uniform meshes directly to equation (1.2), we introduce two transformations which regularise the behaviour of the solution. This idea was discussed by Norbury **[10]** with regard to a particular singular Volterra equation.

If we define

$$s = t\sin^2\phi, \tag{3.1}$$

then equation (1.2) becomes

$$y(t) = g(t) + \int_0^{\pi/2} 2\sqrt{t}\sin\phi\, y(t\sin^2\phi)\,d\phi.$$

Setting

$$t = \theta^2 \tag{3.2}$$

and defining $u(\theta) \equiv y(t)$ and $h(\theta) \equiv g(t)$, we obtain

$$u(\theta) = h(\theta) + \int_0^{\pi/2} 2\theta \sin \phi u(\phi^*)\, d\phi, \tag{3.3}$$

where

$$\phi^* = \theta \sin \phi. \tag{3.4}$$

These two transformations have regularised the behaviour of the solution **[9]** and hence we are in a position to apply standard product integration techniques to solve equation (3.3). We divide the range of θ by the equally spaced mesh points $\theta_j = jh$, $j = 1, \ldots, i$ with a mesh spacing of length h. We then subdivide the range of integration ϕ by the unequally spaced mesh points ϕ_{ij}, $j = 1, \ldots, i$ defined by

$$\sin \phi_{ij} = \theta_j / \theta_i. \tag{3.5}$$

Hence, from equation (3.3) we have

$$u(\theta_i) = h(\theta_i) + \sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} 2\theta_i \sin \phi u(\phi^*)\, d\phi. \tag{3.6}$$

By virtue of the non-uniform division of the range of integration, the unknown in the integrand, $u(\phi^*)$, takes on values $u(\theta_j)$ at the end of each range of integration. We now introduce a trapezium rule approximation to equation (3.6); defining u_j to be our numerical approximation to the true solution $u(\theta_j)$, we obtain

$$u_i = h(\theta_i) + \sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} 2\theta_i \sin \phi \frac{[(\phi_{ij} - \phi)u_{j-1} + (\phi - \phi_{ij-1})u_j]}{(\phi_{ij} - \phi_{ij-1})}\, d\phi. \tag{3.7}$$

The coefficients of the u_j's may be integrated explicitly to yield the expression

$$u_i = h(\theta_i) + 2\theta_i \sum_{j=1}^{i} a_{ij}u_{j-1} + b_{ij}u_j, \tag{3.8}$$

where

$$0 < a_{ij} = \cos \phi_{ij-1} - \left(\frac{\sin \phi_{ij} - \sin \phi_{ij-1}}{\phi_{ij} - \phi_{ij-1}}\right) \tag{3.9}$$

and

$$0 < b_{ij} = \left(\frac{\sin \phi_{ij} - \sin \phi_{ij-1}}{\phi_{ij} - \phi_{ij-1}}\right) - \cos \phi_{ij}. \tag{3.10}$$

THEOREM 3.1. *The solution u_i to the discrete equation* (3.8) *is convergent to the solution $u(\theta_i)$ of the integral equation* (3.8) *with order $h^{\frac{3}{2}-\varepsilon}$, ε an arbitrary positive number.*

The proof of this theorem requires a number of preliminary results which we now prove.

LEMMA 3.1. *With d_{ij}'s defined as in* (3.21), *$h \sum_{j=1}^{i-1} d_{ij}^{(1)}$ is bounded above independently of h.*

Proof. Since $1 - hd_{ii}$ is independent of h, to first order, it suffices to show that

$$h \sum_{j=1}^{i-1} d_{ij} = 2\theta_i \sum_{j=1}^{i-1} (\phi_{ij+1} - \phi_{ij-1})$$

is bounded above independently of h.

We define $p(\theta, \theta_i)$ by

$$p(\theta, \theta_i) = \sin^{-1}\left(\frac{\theta + h}{\theta_i}\right) - \sin^{-1}\left(\frac{\theta - h}{\theta_i}\right).$$

Then from definition (3.5) we see that

$$h \sum_{j=1}^{i-1} d_{ij} = 2\theta_i \sum_{j=1}^{i-1} p(\theta_j, \theta_i). \tag{3.11}$$

Differentiation shows that $p(\theta, \theta_i)$ is an increasing function of θ for $\theta \in [\theta_1, \theta_{i-1})$, and hence that

$$\sum_{j=1}^{i-2} p(\theta_j, \theta_i) \leqq \frac{1}{h} \int_{\theta_1}^{\theta_{i-1}} p(\theta, \theta_i)\, d\theta.$$

On applying the Mean Value Theorem to $p(\theta, \theta_i)$, we obtain

$$\sum_{j=1}^{i-2} p(\theta_j, \theta_i) \leqq \int_{\theta_1}^{\theta_{i-1}} \frac{2\, d\theta}{[\theta_i^2 - (\theta + h)^2]^{\frac{1}{2}}} \leqq \pi.$$

By applying this result to equation (3.11) and noting that hd_{ii-1} is $O(h^{\frac{1}{2}})$, the lemma follows.

LEMMA 3.2. *The second iterated kernel $d_{ij}^{(2)}$ generates a function $f^{(2)}((j+1)h, ih)$ which satisfies the conditions that $f^{(2)}(t, t)$ exists, $\dfrac{\partial f^{(2)}}{\partial s}(s, t) \leqq 0$ and $\dfrac{\partial f^{(2)}}{\partial t}(s, t) \geqq 0$.*

Proof. From definitions (2.8), (3.5) and (3.21) we have for $j = 1, \ldots, i-1$, where M is independent of h,

$$d_{ij}^{(2)} = h \sum_{\ell=j+1}^{i-1} d_{i\ell}^{(1)} d_{\ell j}^{(1)} \leqq \frac{4\theta_i^2 M}{h} \sum_{\ell=j+1}^{i-1} (\phi_{i\ell+1} - \phi_{i\ell-1})(\phi_{\ell j+1} - \phi_{\ell-1}). \tag{3.12}$$

Consequently, we analyse the sum S, defined by

$$S = \sum_{\ell=j+1}^{i-1} (\phi_{i\ell+1} - \phi_{i\ell-1})(\phi_{\ell j+1} - \phi_{\ell j-1}). \tag{3.13}$$

Use of the Mean Value theorem shows that

$$S = \sum_{\ell=j+1}^{i-1} \frac{4h^2}{[\theta_i^2 - A^2]^{\frac{1}{2}}[\theta_\ell^2 - B^2]^{\frac{1}{2}}}$$

where

$$(\theta_\ell - h) < A < (\theta_\ell + h) \leqq \theta_i$$

and

$$(\theta_j - h) < B < (\theta_j + h) \leqq \theta_\ell.$$

We now represent the summation as the sum of two Riemann sums for the purpose of comparison with an integral. We obtain

$$S \leqq 4h \int_{\theta_{j+1}}^{\theta_{i-1}} \frac{d\theta}{[\theta_i^2 - (\theta+h)^2]^{\frac{1}{2}}[\theta^2 - \theta_{j+1}^2]^{\frac{1}{2}}}.$$

Noting that for $\theta_{j+1} \leqq \theta \leqq \theta_{i-1}$, $[\theta_i^2 - (\theta+h)^2]^{-\frac{1}{2}} \leqq [\theta_{i-1}^2 - \theta^2]^{-\frac{1}{2}}$, we obtain

$$S \leqq 4h \int_{\theta_{j+1}}^{\theta_{i-1}} \frac{d\theta}{[\theta_{i-1}^2 - \theta^2]^{\frac{1}{2}}[\theta^2 - \theta_{j+1}^2]^{\frac{1}{2}}}.$$

Substitution of $\theta_{j+1}V = \theta$ yields

$$S \leqq \frac{4h}{\theta_{i-1}} \int_1^C \frac{C\,dC}{(C^2 - V^2)^{\frac{1}{2}}(V^2-1)^{\frac{1}{2}}},$$

where $C = \theta_{i-1}/\theta_{j+1}$. This integral is a complete elliptic integral of the second kind (see [**1**], p. 596 and p. 590) and the result may be written (in the notation of [**1**]) as

$$S \leqq \frac{4h}{\theta_{i-1}} F(\pi/2\backslash\cos^{-1}(1/C))$$

$$= \frac{4h}{\theta_{i-1}} \mathbb{K}(1 - 1/C^2). \tag{3.14}$$

Thus, by (3.12), (3.13) and (3.14) we obtain

$$d_{ij}^{(2)} \leqq \frac{16\theta_i^2 M}{\theta_{i-1}} \mathbb{K}(1 - 1/C^2), \tag{3.15}$$

where $C = \theta_{i-1}/\theta_{j+1}$. Thus $d_{ij}^{(2)} \leqq \bar{M} \,.\, \mathbb{K}(1 - \theta_{j+1}^2/\theta_{i-1}^2)$ where $\bar{M}$ is independent of h. Using the known properties of $\mathbb{K}$ (see the graph in [**1**], p. 592) we can finally bound $d_{ij}^{(2)}$ by

$$d_{ij}^{(2)} \leqq \bar{M} \,.\, \mathbb{K}(1 - \theta_{j+1}^2/\theta_i^2).$$

Hence

$$f^{(2)}(s, t) = \bar{M} \,.\, \mathbb{K}(1 - s^2/t^2). \tag{3.16}$$

Since $\mathbb{K}(0) = \pi/2$ and $\mathbb{K}'(x) \geqq 0$ [**1**], the three conditions of the lemma are satisfied.

Lemma 3.3. *The function $f^{(2)}(s, t)$ defined in* (3.16) *is integrable with respect to s, over the range $0 \leqq s \leqq t$, and the result is bounded independently of h. Throughout this lemma $\mathbb{K}$ is as defined in Lemma* 3.2.

Proof. Define I by

$$I = \int_0^t f^{(2)}(s, t)\, ds.$$

Then, using definition (3.16) and making the substitution $s = Vt$, we obtain

$$I = \bar{M} \int_0^1 \mathbb{K}(1 - V^2) t\, dV.$$

We can then show that I satisfies

$$I \leqq \bar{M}t \int_0^1 \frac{\mathbb{K}(1-V^2)\sqrt{2}\,V\,dV}{[1-(1-V^2)^2]^{\frac{1}{2}}}.$$

Setting $y = 1 - V^2$ yields

$$I \leqq \frac{\bar{M}t}{\sqrt{2}} \int_0^1 \frac{\mathbb{K}(y)}{(1-y^2)^{\frac{1}{2}}}\,dy = \frac{\bar{M}t}{\sqrt{2}}\mathbb{K}^2(1/\sqrt{2}),$$

using [**7**, formula 6.143, p. 637]. Thus we have the desired result.

LEMMA 3.4. *The consistency error, ce_i, satisfies*

$$ce_i \leqq \delta = M_5 h^{\frac{3}{2}-\varepsilon}.$$

Throughout this lemma M_K, $K = 1, \ldots, 5$ denotes a number bounded above independently of h.

Proof. The consistency error ce_i is defined by

$$ce_i = \left|\sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} 2\theta_i \sin\phi \left[u(\phi^*) - \left\{\frac{(\phi_{ij}-\phi)u(\phi_{j-1}) + (\phi - \phi_{ij-1})u(\phi_j)}{(\phi_{ij}-\phi_{ij-1})}\right\}\right] d\phi\right|.$$

By using the formula for the error in a Lagrange interpolating polynomial, we obtain

$$ce_i = \left|\sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} \theta_i \sin\phi \frac{d^2(u(\phi^*))}{d\phi^2} \times (\phi - \phi_{ij-1})(\phi - \phi_{ij})\,d\phi\right|_{\phi^*=\eta}.$$

Because of the smoothness of the solution $u(\theta)$ of equation (3.3) (see [**9**]), we may bound ce_i above by

$$ce_i \leqq \left|\sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} M_1(\phi - \phi_{ij-1})(\phi_{ij} - \phi)\sin\phi\,d\phi\right|.$$

Maximising the quadratic term over (ϕ_{ij-1}, ϕ_{ij}), we obtain

$$\begin{aligned} ce_i &\leqq M_2 \sum_{j=1}^{i} (\phi_{ij} - \phi_{ij-1})^2(\cos\phi_{ij-1} - \cos\phi_{ij}) \\ &= M_2 \sum_{j=1}^{i} G(\theta_j, \theta_i), \end{aligned}$$

where

$$\begin{aligned} G(\theta_j, \theta_i) = &\left[\sin^{-1}\left(\frac{\theta_j}{\theta_i}\right) - \sin^{-1}\left(\frac{\theta_{j-1}}{\theta_i}\right)\right]^2 \\ &\times \left(\left[1 - \frac{\theta_{j-1}^2}{\theta_i^2}\right]^{\frac{1}{2}} - \left[1 - \frac{\theta_j^2}{\theta_i^2}\right]^{\frac{1}{2}}\right). \end{aligned}$$

It can be shown that $G_\theta(\theta, \theta_i) \geqq 0$ for $0 \in [0, \theta_i]$ and so, by an integral comparison theorem

$$ce_i \leqq \frac{M_2}{h} \int_{\theta_1}^{\theta_i} G(\theta, \theta_i)\,d\theta + M_2 G(\theta_i, \theta_i). \tag{3.17}$$

By defining $x=\theta/\theta_i$ and $y=(\theta-h)/\theta_i$ and applying the result that $(A^{\frac{1}{2}}-B^{\frac{1}{2}})<(A-B)^{\frac{1}{2}}$, we obtain

$$J=\int_{\theta_1}^{\theta_i} G(\theta,\,\theta_i)\,d\theta \leqq \int_{\theta_1}^{\theta_i} (x^2-y^2)^{\frac{1}{2}}\,[\sin^{-1}(x)-\sin^{-1}(y)]^2\,d\theta.$$

In the following we use the Mean Value Theorem (twice) and the result that $(A-B)^{1+\varepsilon}\leqq(A^{1+\varepsilon}-B^{1+\varepsilon})$ for all $\varepsilon\geqq 0$.

$$\begin{aligned}[\sin^{-1}(x)-\sin^{-1}(y)]^2 &= [\sin^{-1}(x)-\sin^{-1}(y)]^{1-\varepsilon}[\sin^{-1}(x)-\sin^{-1}(y)]^{1+\varepsilon}\\ &\leqq \frac{(x-y)^{1-\varepsilon}}{(1-x^2)^{\frac{1}{2}-\varepsilon/2}}[\sin^{-1}(x)-\sin^{-1}(y)]^{1+\varepsilon}\\ &\leqq \frac{(x-y)^{1-\varepsilon}}{(1-x^2)^{\frac{1}{2}-\varepsilon/2}}[(\sin^{-1}(x))^{1+\varepsilon}-(\sin^{-1}(y))^{1+\varepsilon}]\\ &\leqq (1+\varepsilon)\left(\frac{\pi}{2}\right)^{\varepsilon}(x-y)^{2-\varepsilon}/(1-x^2)^{1-\varepsilon/2}.\end{aligned}$$

The introduction of $\varepsilon>0$ has made the term $(1-x^2)^{-1+\varepsilon/2}$ integrable. Thus we have

$$\int_{\theta_1}^{\theta_i} G(\theta,\,\theta_i)\,d\theta \leqq M_3\int_{\theta_1}^{\theta_i}\frac{(x^2-y^2)^{\frac{1}{2}}(x-y)^{2-\varepsilon}}{(1-x^2)^{1-\varepsilon/2}}\,d\theta,$$

which is

$$\leqq M_3\int_{\theta_1}^{\theta_i}\frac{(2x)^{\frac{1}{2}}(x-y)^{\frac{5}{2}-\varepsilon}}{(1-x^2)^{1-\varepsilon/2}}\,d\theta.$$

On noting that $(x-y)=h/\theta_i$ we obtain

$$\int_{\theta_1}^{\theta_i} G(\theta,\,\theta_i)\,d\theta \leqq M_4 h^{\frac{5}{2}-\varepsilon}.$$

On noting that $G(\theta_i,\,\theta_i)$ is $O(h^{\frac{3}{2}})$ we obtain, from inequality (3.17),

$$ce_i \leqq M_5 h^{\frac{3}{2}-\varepsilon}, \tag{3.18}$$

the desired result.

Proof of Theorem 3.1. Defining $\varepsilon_i=(u_i-u(\theta_i))$ to be the approximation error at each mesh point, subtraction of equation (3.7) from equation (3.6) gives us

$$\varepsilon_i \leqq \delta+\sum_{j=1}^{i} 2\theta_i(a_{ij}\varepsilon_{j-1}+b_{ij}\varepsilon_j) \tag{3.19}$$

where δ is an upper bound on the consistency error, defined in Lemma 3.4. Defining e_{ij} by

$$e_{i0}=2\theta_i a_{i0},$$

$$e_{ij}=2\theta_i(a_{ij+1}+b_{ij}),\quad j=1,\ldots,i-1$$

and

$$e_{ii}=2\theta_i(b_{ii}),$$

inequality (3.19) may be written as

$$\varepsilon_i \leqq \delta + \sum_{j=0}^{i} e_{ij}\varepsilon_j.$$

For $j = 1, \ldots, i-1$ it is possible to show, by use of the Mean Value Theorem, that

$$e_{ij} \leqq 2\theta_i(\phi_{ij+1} - \phi_{ij-1}).$$

Assuming that $\varepsilon_0 = 0$, since the initial value is known, we may write

$$\varepsilon_i \leqq \delta + h \sum_{j=1}^{i} d_{ij}\varepsilon_j, \tag{3.20}$$

where

$$hd_{ii} = 2\theta_i b_{ii}$$

and

$$hd_{ij} = 2\theta_i(\phi_{ij+1} - \phi_{ij-1}), \quad \text{for} \quad j = 1, \ldots, i-1. \tag{3.21}$$

This is in the form of inequality (2.5) and hence we attempt to apply Theorem 2.2. Analysis of b_{ii} shows it to be $O(h^{\frac{1}{2}})$ and thus condition (i) is satisfied for sufficiently small h. Lemmas 3.1, 3.2 and 3.3 show us, respectively, that conditions (ii), (iii) and (iv) of Theorem 2.2 are satisfied. Hence we deduce from (3.12) that

$$\varepsilon_i \leqq c_1\delta,$$

where c_1 is independent of h. By Lemma 3.4 we obtain the final result

$$\varepsilon_i \leqq c_2 h^{\frac{3}{2}-\varepsilon}, \tag{3.22}$$

where c_2 is also independent of h.

4. Conclusions

We have derived a set of conditions on discrete error equations which ensure that the rate of convergence of a numerical method will be determined entirely by the consistency error (and for more general higher order methods by the starting values as well). We have shown, by use of a simple linear integral equation (1.2) that these conditions are of practical value in proving convergence. The choice of the simple form of equation (1.2) was made purely to clarify the analysis. The numerical method and proof of convergence would apply equally well to equations with more general kernels of the form

$$y(t) = g(t) + \int_0^t \frac{K(t, s)y(s)}{(t-s)^{\frac{1}{2}}}\, ds. \tag{4.1}$$

Furthermore, under suitable conditions on the behaviour of the kernel K, the basic idea of the numerical method extends to equations of the general form, for $0 < \alpha < 1$,

$$y(t) = g(t) + \int_0^t \frac{K(t, s, y))}{(t-s)^{\alpha}}\, ds. \tag{4.2}$$

We consider this type of equation in Part II of the paper, see **[11]**.

Finally we note that we leave open the following two questions. Is the ε arising in the $O(h^{\frac{3}{2}-\varepsilon})$ rate of convergence an indication that the method cannot attain $O(h^{\frac{3}{2}})$ convergence (it might be $O(h^{\frac{3}{2}} \operatorname{Ln} h)$ for example), or is the ε a product of the analysis of the consistency error resulting, perhaps, from the repeated use of the Mean Value Theorem in Lemma 3.4? What rate of convergence would follow from higher order integration rules being used to approximate the transformed equation (3.3)? In practice we recommend use of the trapezium rule because of its simplicity and its stability properties (see Part II of this paper [**11**]).

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